Children's Awareness of Structural Relationships Embedded in Addition and Subtraction Word Problems

Thesis

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Abstract

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A common practice in primary schools is the use of tasks that involve children solving word problems. A great deal of research has focused on improving children's performance in these tasks by various forms of instruction. This study aims to reveal transitions in children's thinking about; and awareness of structural relationships embedded in word problems by involving children in tasks focused on word problems, but getting them to sort, and match problems, and to construct their own similar problems. The intention was to investigate the nature of the activity that learners engage in when working on these tasks with a view to informing discussion about the role of word problems in the mathematics curriculum.

A small group of Year 4 children worked on solving, sorting, matching and constructing tasks approximately once every four weeks over a several month period. Data was collected in the form of written recordings made by the children, transcripts of dialogue and field notes. The approach taken to the study was broadly in the spirit of grounded theory and drew heavily on the Discipline of Noticing (Mason, 2002). Using aspects of this framework, accounts-of the data were constructed and these were carefully analysed with the intention of providing accounts-for the phenomena that were observed.

Two main themes emerged from the analysis. The first I describe as persistent behaviour. This is the tendency for children to display similar behaviour and attend to similar foci across tasks and over time, despite this behaviour, on occasion, being inappropriate. The second theme, in contrast, was unstable behaviour. This is the tendency for children to display variation in behaviour when responding to tasks that might be expected to produce similar responses. Attempts are made to account-for these phenomena and implications for teaching and learning are discussed. It is concluded that the potential of word problems as a pedagogical tool might be better exploited if learners experienced long term immersion in sorting, matching and constructing tasks in addition to the more common focus on solving.

1.0 Introduction

1.0 Introduction

My motivation for this study arose out of my experiences as a school pupil, as a teacher, as a teacher educator and also out of my reading and conversations with others about mathematics education. I have become particularly interested in that area of the primary curriculum that is identified as "using and applying mathematics" and embarked on the research with the (somewhat naive) intention of finding out something which might benefit pupil performance in this area. I worked on a regular basis with a small group of primary school children focusing on tasks involving the use of word-problems. I feel that I have discovered much about the nature of the activity that they engage in when working on these tasks and may also have provided some stimulus in moving them on in their learning. However I have also come to realise that I have learnt much about myself and have developed greatly as a result of my engagement with this study. So although the study is about the behaviour of a group of children working on mathematics tasks, it is also about my development and the effects that the whole process has had on me. With this in mind I will begin by giving an account of those experiences that I feel have particularly influenced my interest and shaped the sensitivities that I brought to the study. I will then outline more specifically the questions that I set out to answer.

1.1 Learning in Primary School

My main recollections of primary school mathematics are of doing "sums and problems". Sums were not of course just sums but differences, products and quotients as well. Problems were 'sums' dressed up in words. I don't remember much about how I was taught but I was not aware of lesson objectives being made clear to me at the beginning of lessons or of any discussion of tasks being allowed. I was exposed to standard written algorithms and remember having lots of practice doing them. We worked on squared paper and calculations were required to be set out in a prescribed manner. We completed many pages of them. I generally managed to get calculations correct and tended not to be bored by this repetitive approach. I don't remember being encouraged to perform calculations in ways other than these standard methods. However, I do remember being flexible in the way that I tackled these calculations. Although I set out calculations in the prescribed manner my thinking about them differed depending on the numbers involved. This was usually the case when there was the possibility of a 'shortcut' that made the process easier. For example when multiplying by 15 I would begin by multiplying by 10 (possibly by adding a zero!) and then halving the answer to find the result to multiplying by 5. I would then write these products in the appropriate format and then add them to find the answer to the overall calculation. This was not the standard method that I had been taught but it worked well for multiplication calculations involving the number 15. In a similar fashion I developed methods for other particular numbers or combinations of numbers. I can remember starting 'calculation practice' (such were maths lessons in many of my primary school classes) by looking for the calculations I could do using my special methods. As far as I am aware my teacher thought that I was doing the calculation in the way I had been taught and wasn't aware of my thinking as this didn't show up in the recording of my work.

I don't remember any teaching about how to tackle problems in my primary classes; I was just given them to do. It seemed fairly obvious to spot the calculations that were required. Perhaps I relied on clues that didn't involve actually engaging deeply with the problem such as those presented by Sowder (1988) which are discussed later. We were taught to set out problems though and all of mine were carefully laid out with an accompanying calculation and the answer written in a sentence. Although I have some quite vivid recollections of my own performance in primary mathematics, or arithmetic as our books were titled, I don't remember much about how my peers performed but I know from my parents that I was neither particularly good nor particularly poor in relation to my classmates.

1.2 Teaching in Primary School

Although I didn't realise it at the time, as a learner I was adopting strategies that I subsequently hoped that children in my classes would take when I became a teacher. It seemed to me quite normal and natural that primary school learners should 'see' and use the strategies that I had seen and used. Of course a significant proportion of children in my classes didn't develop such strategies and, at least in relation to mathematics, didn't seem to be able to think for themselves, wanting or needing to be told exactly what to do in every new situation they encountered. My efforts at encouraging many of the children to develop flexible strategies seemed to make life harder for many of them as they seemed to see these strategies as different *standard methods* to be used in particular situations, not 'shortcuts' to be used flexibly. They didn't understand these strategies any more than they understood the standard written algorithms and generally forgot them or at best confused elements of different approaches. A particularly worrying incident was watching a child perform the calculation 400 – 1 by the decomposition method. Even more worrying was that after she had laboriously negotiated the complex re-grouping operations and wrote down the answer of 399 she expressed no indication whatsoever that she was aware that there might have been an easier way of doing things, even after this was explained to her!

I wasn't any more successful teaching word problems, being very worried at my inability to help children to choose the right operation to solve a problem. I developed a strategy for pupils to use that was not dissimilar to the one I had been encouraged to use as a young learner. The idea was that pupils should apply a particular framework to their thinking and to the format of their written responses. The framework applied to problems was '*information- calculation- solution*'. This being a prompt to remind children that first they had to identify the relevant information in the problem before carrying out the appropriate calculation and finally writing the answer or solution in a sentence. I subsequently replaced this framework with various versions of more detailed 'answer frames' (an example of an 'answer frame' can be found in Appendix 1). These are usually in the form of single sheets of paper divided into several sections. Each section contains a prompt for the child to follow when tackling a problem and a space in which to record something relating to the prompt. Prompts might focus on identifying the relevant and irrelevant information, recording the required calculation, writing the answer and so on. These approaches met with some success. Children who previously found word problems very difficult at least wrote something. The written responses to problems looked good but I'm not sure that I did anything that really enhanced the pupils' understanding of word problems. I really couldn't see any way of helping them through what I now consider to be the most important stage in solving word problems, deciding on what mathematics to use. In particular I was concerned that much of what I did with my pupils involved a mechanistic jumping through hoops rather then the creation of any real understanding. This seemed to be reinforced by the nature of some of the material involving problems that I had access to. One book, 'Target Maths' I think it was called, began with a page entitled 'Addition Problems'. Each problem contained two numbers and the correct answer could be found by adding these numbers. This didn't so much result in mechanistic thinking by my pupils, but perhaps no real thinking at all. To counter the effects of such material I would frequently write my own problems that contained redundant information and mixed 'adds' in with 'takeaways' and other types of problem. I realised that this was a more useful range of problems to use but still found myself lacking in finding ways of helping the pupils to develop strategies to address them.

1.3 Personal Development and Teacher Education

As a result of the interest that I developed in mathematics arising out of teaching the subject I embarked on what we would now call a programme of personal development. I didn't see it in this way at the time and certainly had no pressure exerted on me by my managers. I studied various courses about mathematics and mathematics education because they interested me. I didn't know it at the time but my actions were an example of what I have come to know as *asserting* behaviour, an important response to tasks and situations that will feature later in the discussion. Subsequently I took a position as a lecturer in mathematics education working with primary undergraduate and primary PGCE trainees. This has been extremely interesting and rewarding but also very thought provoking. The aspect of my experiences that has been particularly striking has been the observation that many of my trainees have displayed precisely the same characteristics that I have encountered with the primary school children that I taught. This is despite the requirement that all trainees need to acquire a grade C at GCSE Mathematics in order to be accepted on the course. Mathematics subject knowledge is a concern with many trainees but what has worried me more is the tendency of many to want to be told what to do rather than to think for themselves. This has been particularly evident in early attempts to work with them on investigations where many students display a very limited range of strategies. Some students seem to have little idea about how to proceed and in my view have under-developed 'using and applying' skills. It is worth mentioning that in some cases a fear of doing things wrongly seemed to influence the student performance that I observed. These observations have been consistent over a decade of cohorts.

1.4 The Study

A theme which runs through my experiences as a primary school teacher and as an ITT lecturer is this relative lack of initiative evident in a significant number of learners, and the tendency for them not to see things that appeared clear and obvious to me, even at a young age. Similar observations have been evident in conversations with colleagues and reported in literature that I have read, further fuelling my interest. One particularly interesting book, *Numeracy and Beyond* by

Hughes, Desforges, and Mitchell (2000) which considers the issue of application in some depth, contains a sentence that crystallised in my mind the phenomenon that I was interested in:

> "There is little point in pupils being 'numerate' if they cannot apply what they know."
>
> (Hughes et al., 2000, page 2)

This quote of course opens up discussion of what actually is meant by the terms *numerate*, *apply* and *know*. However these terms will not be explored here as discussion would be lengthy. In place of such a discussion an account from the book may be useful to clarify the point being made.

> 'It was found that 80% of 12 year-olds could quickly and correctly divide 225 by 15. However, only 40% of the same sample could solve the problem 'if a gardener had 225 bulbs to place equally in 15 flower beds, how many would be put in each bed?' Most of the failing pupils did not know which mathematical procedure to use, although they were capable of conducting the routine once the appropriate operation was named.'
>
> (Hughes et al., 2000, page 7)

In the example many children appear to be able to carry out a division calculation but do not seem to see the connection between this specific calculation and the relatively straight forward problem that is described. This account concerned and interested me and along with the preceding quote became the stimulus and starting point for my study. I became interested in how I could help learners *apply* the mathematics to which they had been exposed. I wanted to investigate the area of using and applying with the intention of discovering something that would improve pupil performance. In particular I hoped that I might discover something that would help me to generate independence and initiative in learners when they encountered novel situations. I wanted to find out how I might encourage them to become flexible thinkers and to *see* aspects of mathematics in ways similar to how I had seen them.

Initially I was reluctant to focus my study on word problems as a significant quantity of material on this area was already evident in the literature. I wanted to examine some more authentic situations but the schools that I had access to were wary of any intervention that involved a significant diversion from their normal curriculum. Thus I focused on the idea of working with word problems as the aspect of 'using and applying' which was most evident in the school to which I had access. However in addition to giving children problems to solve, I used tasks that involved sorting, matching and constructing word problems. These situations were novel to the children that I was working with. I hoped that collecting and analysing data from sessions involving these tasks would help me to address my initial general research questions which were:

> How does a learner proceed in addressing a word problem situation?
>
> What can teachers do to help children in these situations?

I carried out a pilot study using word problem sorting, matching and constructing tasks with a small group of learners which caused me to develop these questions. In particular I became interested in children's awareness of structural relationships embedded in word problems and how their thinking about these relationships might change over time and with exposure to tasks involving sorting, matching and constructing word problems. To reveal useful data about this the main focus of my study became to address the following questions:

What is the nature of the activity that learners engage in when responding to tasks that involve:

- solving word problems,
- sorting word problems,
- matching word problems;
- constructing word problems

How can phenomena identified when analysing children's activity on these tasks be accounted-for in terms of relevant literature?

What are the implications for teaching and learning arising from this analysis?

A focus on my own development was not to the fore of my mind at this stage of the study so I did not have a specific question relating to this. However this did emerge as an important aspect during the study so reference will be made to my own development as appropriate during the thesis.

The next chapter will further set the scene for the study. Appropriate literature relating specifically to word problems will be reviewed and reference will be made to other relevant material from the domain of mathematics education. Chapter 3 will then describe the conduct of the study in more detail and provide a rationale for the decisions made.

2.0 Setting the study in context

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A broad aim of this study is to inform the teaching and learning of mathematics focusing on the use of tasks that involve word problems. Thus it is appropriate to review material relating to the role of word problems in the curriculum and to children's responses to these tasks. It is also appropriate to present a discussion of relevant ideas about the teaching and learning of mathematics in general. This chapter begins with a discussion of what is meant by the terms 'problem' and 'word problem'. Particular structures and categories of word problem will be identified. The discussion then moves to examine the role of word problems in the curriculum and children's responses to the typical word problem tasks evident in this. To conclude this discussion of word problems, two mental models that attempt to account-for the potential stages in problem solving are reviewed. The latter part of the chapter reviews some more general ideas about teaching and learning that initially seem to have particular relevance to this study.

2.1 What is a (word) problem?

This study focuses on primary school children working with word problems. From my experience as a learner and a teacher, word problems seem to have a prominent position in the primary mathematics curriculum. However, until relatively recently, I had worked with problems without considering too deeply the meaning of the word. For me the word 'problem' had been defined by its usage in the mathematics curriculum and in particular in mathematics text books. A common approach in such material is to present an exercise of calculations followed by a series of word problems. These problems usually described situations intended to be recognisable to pupils and often involved the use of operations related to those used in the preceding calculation exercise. The goal of working on a problem always seemed to be to produce a correct answer. Thus for me a problem became defined as a situation described in words that required me to use some mathematics in order to produce an answer. It didn't matter whether the answer to the problem was obvious or if it required considerable thought, if a task was presented in 'word problem' form then it was a problem. In line with this, tasks that were presented as calculations were not problems; they were calculations even if the way forward was not obvious and required much thought. Such was the effect of the usage of the word 'problem' in the culture of the mathematics classroom I was exposed to, that I still find myself using 'problem' in this way today and have to consciously monitor my speech to ensure that I use the word in line with the more precise meaning that I have come to understand.

Before discussing this particular meaning it is worth highlighting that even in current documentation presented by the Primary National Strategy (PNS) the meaning of the word is to some extent taken for granted and used in a way that is consistent with my initial understanding. In the PNS publication "*Using and Applying Mathematics – Guidance Paper*" (DCSF, 2006a) there seems no attempt to define the word whilst maintaining that problem-solving in the Primary Framework 'focuses on problems that involve calculations set in wider-ranging contexts..." (DCSF, 2006a, page 9). Examples of problems in this document also reflect the view of the word that I developed from experience. I now have a different view of the type of situation that constitutes a problem which is appropriately reflected by Haylock (2006) who explains:

> "A problem, as opposed to something that is merely an exercise for practising a mathematical skill, is a situation in which we have some givens and we have a goal, but the route from the givens to the goal is not immediately apparent."
>
> (Haylock , 2006, page 317)

This view is an echo of that expressed by other authors such as Skemp (1993) who explains that the essence of a problem is "that we want to do something but do not know how." Wiliam (1998) goes further suggesting that the ideas of intent and obstacles are important explaining that "a problem is not a problem if you don't want to solve it" and that "if you know what to do then there's no problem." (Wiliam, 1998, page 1)

Applying the idea of a problem having to involve an obstacle to my early experience, the word problems that I encountered were only 'problems' if the route to the solution was not immediately clear to me. Thus many of the word problems that I worked with, and many that learners in my classes worked with, were not problems because the way forward to the solution in many cases *was* immediately obvious. Also, as Haylock points out, what may be a clear situation for one learner to address may be a problem for another. Furthermore a mathematical task need not be set in a described context to be a problem. Some of the calculations that I encountered as a learner were problems for me as I could not immediately see the route to the solution. The defining feature of a problem is then in the 'eyes of the learner', if a learner does not immediately see the route forward from a set of givens to a solution then the situation is a problem.

How then does a learner proceed in addressing a problem situation if the way forward is not immediately obvious? What can teachers do to help children in such situations? The first of these questions reflects the initial focus of this study while the second represents my aim of endeavouring to generate some suggested approaches that may be further explored in daily activity in primary classrooms. However my focus is not on addressing or working with any problem situation likely to arise in the mathematics classroom but on a narrower and more manageable area of tasks that were previously identified as word problems. So what is a word problem?

Verschaffel, Greer and de Corte (2000) explain that word problems are also referred to as "verbal problems" or "story problems" and describe the difficulties in devising a 'precise and complete definition' of the term. Of course one of the issues in providing any clarification of the terms is that use of the word "problem" may not be appropriate depending on the learner considered. However, neglecting this anomaly for the time being Verschaffel, Greer and de Corte (2000) define word problems as:

> "Verbal descriptions of problem situations wherein one or more questions are raised the answer to which can be obtained by the application of mathematical operations to numerical data available in the problem statement"
>
> (Verschaffel et al., 2000, page ix)

They further illustrate this by identifying that "a characteristic feature of word problems is the use of words to describe a (usually hypothetical) situation" (page ix). So using their examples the problem statement:

> "Pete wins 3 marbles in a game and now has 8 marbles.
> How many marbles did he have before the game?"
>
> (Verschaffel et al., 2000, page ix)

is a word problem whereas "8 — 3 = ?" is not. Although the latter number statement may present difficulties to some learners that cause it to be a 'problem' to them, it is excluded from the category of word problems. Tasks such as "What do you get if you subtract 3 from 8?" are also not considered as being word problems as they do not refer to an "existent or imaginable meaningful context', so verbally stated calculations are not considered to be word problems. In this study word problems are presented in written form but it is acknowledged that such problems presented along with tables, pictures and diagrams or presented verbally still fall within the same category of problem. Verschaffel et al. (2000) do distinguish word problems, as described above and used in school, as a different type of situation to the "quantitative tasks and dilemmas that arise in real-life out-of-school situations". Some authors see this discontinuity as severely limiting any possible value of using word problems in the mathematics classroom. This view will be discussed in more detail later in the chapter when considering the role and value of word problems in the curriculum. Attention now will turn to examining the types of word problem that children are likely to encounter in the primary classroom.

2.2 Structures and categories of word problem

"Word problems" do not seem to be explicitly referred to in the Using and Applying Strand of the PNS Mathematics Framework although the word 'problem' does feature in learning objectives as early as the Foundation Stage. It is clear from the Learning Overviews that many of the 'problem' situations that children encounter are intended to be practical in nature, as the following example from a Year 1 Learning Overview illustrates:

> There are seven beads in this pot. I am putting one more bead in the pot. How many are in there now? How did you know? How can you check?
>
> This time there are ten beads in the pot. I take out one bead. How many beads are left in the pot? How did you know? How can you check?
>
> Start with a different number of beads in the pot. Ask your partner to put another bead in or take one out and then say how many there are in the pot. How will you know if your partner is right?
>
> (DCSF, 2006b)

However by Year 2 children are expected to be working towards the learning objective:

> Solve problems involving addition, subtraction, multiplication or division in contexts of numbers, measures or pounds and pence
>
> (DCSF, 2006c)

Included among the types of situation that teachers are given as examples to use in Assessment for Learning are the following:

> Mina and Ben play a game. Mina scores 70 points. Ben scores 42 points. How many more points does Mina score than Ben?
>
> Rosie spent 48p. Suzy spent 36p more than Rosie. How much did Suzy spend?
>
> (DCSF, 2006d)

These very clearly fit in with the sort of situation that Verschaffel et al. (2000) identify as word problems. So in Year 2 teachers are informed that children are expected to solve word problems that may involve addition, subtraction, multiplication or division. The types of imagined context that are intended to be used are implied by the learning objective and by examples in the learning overviews but there is little guidance about the mathematical structure of the problem statements that children should encounter. To explore the possible range of mathematical structures that may be represented in a problem statement we need to look beyond the support of the PNS Mathematics Framework. For the purposes of this study it is useful to narrow the focus to look only at the mathematical structures that relate to word problems that involve the use of addition or subtraction.

Haylock (2006) describes two different structures of situation that can be modelled by addition. He identifies aggregation as a “situation in which two (or more) quantities are combined into a single quantity and the operation of addition is used to determine the total”. Another way of describing this is that it is “the union of two discrete sets”. The other addition structure is augmentation. This refers to a “situation where a quantity is increased by some amount and the operation of addition is required to find the augmented or increased value.” He explains that money and time (age) provide a frequently used context for this structure, for example:

> “You are 6 years old now. How old will you be in 4 years time?”
>
> (Haylock, 2006, page 32)

Haylock (2006) goes on to outline three structures relevant to subtraction. These are the partitioning, reduction and comparison structures. Partitioning is that situation in which a “quantity is partitioned off in some way... ... and subtraction is required to calculate how many or how much remains”. The reduction structure is similar though it refers to situations in which “a quantity is reduced by some amount and the operation of subtraction is required to find the reduced value." It could be considered as the inverse process of the augmentation structure of addition. The commonly used vocabulary of "take away" is relevant to each of these structures but this is not the case for the third structure. Comparison structures relate to those situations that involve finding the *difference* between two quantities. Unlike the first two structures the physical act of taking away is not relevant to the structure which is reflected in situations like:

> "I am 62 and my daughter is 35. How many years younger than me is she?"

and

> "If holiday package A costs £716 and package B costs £589, how much more expensive is package A?"

There is one further structure relating to addition and subtraction operations. This is the inverse structure in which subtraction can be thought of as the "inverse of addition" and similarly addition can be thought of as the "inverse of subtraction". Authors such as Anghileri (2006) have suggested that learners in the primary school should experience problems reflecting a wide range of such structures and that these should be used as the basis for discussion. So far seven structures of addition and subtraction have been outlined but a larger range of situations results when these structures are represented in word problem statements.

Riley, Greeno and Heller (1983) presented a scheme of classification that involved three types of word problems. *Change* problems referred to 'dynamic situations in which some event changes the value of a quantity". This type of problem includes some sort of action, for example giving, receiving, paying, and in the context of this study getting on or off a bus. Change problems can be subdivided into those that refer to an increasing situation and those referring to decreasing situations. So

> Joe had 3 marbles; then Tom gave him 5 more marbles; how many marbles does Joe have now?

is an example of an increasing situation whereas:

> Joe had 8 marbles; then he gave 5 marbles to Tom; how many marbles does Joe have now?

describes a decreasing situation. It is tempting to label these situations as addition and subtraction respectively. Indeed addition can be used to solve the first problem and subtraction the second. However further analysis reveals that in each case there are three possibilities for the unknown quantity that a problem solver may need to find. The unknown could be the result set as in each of the examples, or it could be the start set or the change set. This would suggest six different structures of change problem. (Please see Appendix 2 for examples of change problems).

In contrast to change problems another category, *combine* problems relate to "static situations involving two amounts". In such cases there is no apparent action taking place. For example in the combine problem:

> Joe has 3 marbles: Tom has 5 marbles; how many marbles do they have altogether?

there is no suggestion that the quantities are actually being put together in a physical sense nor does either child's amount increase. Taking account of the unknown in combine problems suggests that there may be two types of combine problem, those like the example that have the superset as the unknown and those that require the solver to find a subset. (Please see Appendix 3 for examples of combine problems).

Riley et al's. (1983) third category contains compare problems. These involve "two amounts that are compared and the difference between them". The unknown in any compare problem may be the difference, compared or reference set resulting in six variations on this structure of problem. (Please see Appendix 4 for examples of compare problems).

This analysis provides more detail than Haylock's discussion of addition and subtraction through there is a difference in the way that the two situations consider dynamic and static situations. *Change* problems (Riley et al., 1983) are referred to as dynamic and seem to reflect Haylock's augmentation and reduction structures. *Combine* problems (Riley et al., 1983) are identified as static. These situations seem to reflect Haylock's aggregation and partitioning structures. However aggregation and partitioning structures can reflect dynamic *and* static situations. To illustrate this consider the problems below:

> Joe has 3 marbles; Tom has 5 marbles; how many marbles do they have altogether?
>
> Fred has 3 marbles in one hand and 5 marbles in the other; he puts all of the marbles into a single bag. How many marbles are in the bag?

The problems are each *combine* situations and reflect an aggregation structure. However the first problem statement describes a static situation while the second describes a dynamic one. The description of dynamic and static situations in a problem statement could be a feature that influences a learner's response to a problem. There are also other features of problems such as the presence of redundant information and the order of mention of variables which adds to the possible variation of problem types a learner may encounter. This analysis along with Anghileri's advice implies that learners should have exposure to all fourteen types of problem outlined by Riley et al. (1983) as well as experience of these other features.

All word problems seem to share several common features; Verschaffel et al. (2000) describe some of these. They refer to the *format* as the way in which the problem is presented. This would include the nature and complexity of the language used. The *context* is what the problem is about while the *semantic* *structure* is 'the way in which an interpretation of the text points to particular mathematical relationships". The latter may include whether a problem is a change, combine or compare structure. The *mathematical structure* relates to the nature of the given and unknown quantities and the mathematical operation that may be used to solve the problem. Some of these features, like the format, are obvious and explicit and may be considered to be surface features. Others like the semantic structure are less explicit though arguably very important. The semantic structure of the problem can have a significant effect on the difficulty of a problem. For example, in a study by Pauwels (1987) (reported in De Corte and Verschaffel, 1991) 97% of children answered Change 1 type problems correctly whilst 38% provided the correct solution to Compare 6 problems suggesting different cognitive demands being made on learners by the respective semantic structures.

Vergnaud (1982) contributes to discussion about the structure and demands of addition and subtraction word problems by presenting a classification of problems based on cognitive demand. This is done with the intention of developing understanding of the difficulties that learners may display when working with problems. His argument begins by considering the responses of learners to the following three problems:

> **Problem A** : There are 4 boys and 7 girls around a table. How many children are there altogether?
>
> **Problem B**: John just spent 4 francs. He now has 7 francs in his pocket. How many did he have before?

Problem C: Robert played two games of marbles. On the first game, he lost 4 marbles. He played the second game. Altogether, he now has won 7 marbles. What happened in the second game?

He suggests that although the operation 4 + 7 is needed in each case, "Problem B is solved 1 or 2 years later than A, and C is failed by 75% of 11 year old students". In accounting-for this he suggests that there must be some "logical or mathematical difficulties" in Problems B and C that are not evident in Problem A. In an attempt to reveal what these differences might be he introduces the ideas of 'numerical calculus' and 'relational calculus'. The former relates to the operations of addition, subtraction, multiplication and division while by the latter he means "the operations of thought that are necessary to handle the relationships involved in the situation". The 'theorems, assumptions and inferences' that students draw on in this 'relational calculus' he considers are "not necessarily expressed or explained by the children; they can only be hypothesised by observing the children's actions". He calls these "theorems in action" or "inferences in action". (This is an idea rather similar to Mason's *awareness-in-action* which is discussed later in this chapter.)

Vergnaud suggests that children's 'theorems in action' may focus on aspects that are not usually taken account of by the authors of problems, citing time and dimension as examples. He illustrates this by considering Problem D and Problem E below.

Problem D: 4 boys, 7 girls; how many children are there altogether?

Problem E: John has just spent 4 francs, he now has 7 francs; how much did he have before?

Problem D he explains is a measure-measure-measure relationship while Problem E is a measure-transformation-measure relationship and suggests that it is essential to base a classification of problems on a framework developed from this 'relational calculus' rather than on the concepts of addition and subtraction. He goes on to outline six basic categories of relationship he has found evident in addition and subtraction problems. (A summary of this can be found in Appendix 9). Following his experimental analysis of children's responses to these different categories he recommends that research in mathematics teaching should pay more attention to:

- What are the easiest "theorems in action" used by students in solving verbal problems?
- How should we get students to build new theorems by presenting them with new situations?
- How and with the help of which symbolic systems should we help students to make these theorems explicit?
- How can we make sure that theoretical theorems actually become theorems-in-action?

(Vergnaud, 1982, page 59)

It is hoped that this study may contribute to the discussion that these questions provoke.

2.3 Word problems in the Primary National Strategy

The PNS Maths Framework Using and Applying Guidance Paper (DCSF, 2006a) defines numeracy as "a proficiency that requires an inclination and ability to solve number problems in a variety of contexts" and advises that "children need to solve problems to become problem solvers". It is strongly suggested that problem solving is embedded in the teaching and learning of mathematics and teachers are 'warned' that the development of combined "linked chains of calculations, decisions, reasoning and communication" in pupils will "require practice and takes time to learn". It seems to be suggested that learners *work through* "sets of similar questions, puzzles and problems" in order to "recognise how to solve problems of a particular type". Successful learning is seen as children using the 'chains' of reasoning they recognise and becoming able to build their own in order to find out answers to problems. The non-routine nature of using and applying is recognised in this section of the guidance. Various commonly used structured approaches to problem solving are described as useful checklists, but it is noted that these are not always useful in helping children to select an appropriate strategy. The need for careful "selection and dissection of questions, problems and tasks" is recommended in order to provide children with "practice so that they can develop the confidence they need to sustain what can often be a struggle to find a solution."

This of course is guidance and not necessarily the actual practice that takes place in school. It is beyond the scope of this study to provide an accurate overview of problem solving activity in primary schools but one constant theme of my experiences is the observation that teachers tend to use checklists in a fairly prescriptive way. Another interesting observation is the advice given by the PNS (DCSF, 2006e) on planning. Rather than being 'embedded' this guidance suggests that 'using and applying' (and hence problem solving as part of this) has its particular place in the teaching and learning cycle:

> Assess-plan-teach-practice-apply-review
>
> (DSCF, 2006e)

This would seem to suggest that children should first be taught a particular piece of mathematics, practice this mathematics and then solve problems about it. In fact an idea that is prevalent in the guidance is that children should *solve* problems. They are encouraged to read them carefully dissecting them and discussing them but the goal seems to be to develop skills and knowledge to produce correct answers. It is a purpose of this study to consider if word problems can be used in other ways and for other purposes. The next subsection presents a brief review of some of the uses that word problems have been or might be put to, while the subsequent subsection presents the views of some authors who are very critical of the place of word problems in the mathematics curriculum.

2.4 Word Problems: Use beyond the Primary National Strategy

Mason (2001) provides a brief review of the use of word problems through history and across cultures, highlighting effective use and indicating some practices that he sees as "unactualised potential if not actual abuse". He summarises a variety of ways in which word problems have been used including:

- To show off the arithmetic skills of the problem poser;
- To show how a technique is used to solve a class of problems;
- To provide a context so that the solver can more readily locate the required calculations;
- To provide cultural information about what authors have assumed is familiar, interesting, or relevant to their students;
- To induct students into a longstanding cultural practice of classic puzzles;
- To challenge students to think more deeply than just at the level of arithmetical operations;
- As a recreation, like crossword puzzles, and as sheer playfulness;

(Mason, 2001, page 430)

Mason sees making students aware of generality and helping them to generalise for themselves as a central issue in teaching. In supporting this he quotes Zhoubi Suanjing, in a treatise in Chinese from the first century BC, who states " If you can't generalise you have not learnt well enough...". Word problems, Mason suggests, have a role to play in developing the awareness of generality and suggests that as "vehicles for expressing generality" word problems "represent a rich opportunity for exposing students to the importance and significance of algebra." He suggests that we can attempt to realise the pedagogical potential of word problems using a traditional word problem as a starting point and inviting students to:

- Pose another one like it, (in context and numbers), a peculiar one like it (using numbers no-one else is likely to think of), and a general one;
- Pose another one using the same numbers and operations but in a different context, or another problem in the same context but calling upon different operations;
- Discuss different contexts in which the same mathematical problem could arise authentically;
- Consider what values would make a problem realistic, and what additional factors would need to be taken into account to make it more realistic;
- Vary the problem structure by giving different information;
- Generalise the problem by replacing numbers by parameters, and by extending the number of people or objects involved and perhaps generalising to an unspecified number;

(Mason, 2001, p. 435)

These suggestions seem in contrast to the approach advocated by the PNS which as the previous section indicates tends to focus only on solving problems. In fact Mason sees word problems not as an end in themselves but as a pedagogical tool for developing the idea of generality and leading to algebra. The idea of using word problems as a tool for developing understanding of a different area of mathematics is not restricted to Mason. Another example of this follows.

Carpenter and Moser (1982) suggest that there is an implicit assumption in many approaches to school mathematics that "addition and subtraction are best introduced through physical or pictorial representations of joining sets together" (page 9). This assumption is still evident in the current maths curriculum in English schools. For example the following expectation is presented a learning overview from the PNS Mathematics planning for Year 1:

> As they develop their **understanding of addition and subtraction**, children find the number that is one more or one less than a given number practically by adding another object or removing one object from a set then counting the new number. They use their knowledge of the counting sequence and number tracks to predict what number is **one more or one less** than a given number before checking using practical equipment.
>
> (DCSF, 2006b)

There are many such references to the use of practical materials and pictorial representations for developing the ideas of addition and subtraction in the PNS Mathematics materials and in material produced by other authors to support teachers' use of the PNS. A particular example of this is the Abacus Evolve Scheme (Merttens & Kirkby, 2007) where in the Year 1 Block B1 weekly plans there is much reference to counting objects and using pictorial representation to introduce the ideas of addition and subtraction.

Carpenter and Moser (1982) also highlight another implicit assumption in school mathematics. This is that "verbal problems are difficult for children of all ages, and children must master addition and subtraction operations before they can solve even simple verbal problems". This assumption is reflected in the PNS guidance discussed in the previous section which suggests that *teaching* and *practice* should come before *applying*. Carpenter and Moser go on to explain that there is increasing evidence to suggest that these assumptions may be false and provide evidence that many young children can successfully solve basic addition and subtraction problems before they experience formal teaching of addition and subtraction. An argument is then made that, rather then being a task that follows the teaching of addition and subtraction, word problems can provide a "viable alternative for developing addition and subtraction concepts in school".

2.5 A case against word problems

The previous discussion generally has the underlying assumption that word problems are a useful component of mathematics education, that they play an important role in teaching, learning and testing and that a goal of mathematics education is to enable learners to become more proficient in solving them. There is also the assumption that the 'chains of reasoning' that children develop when working on word problems are able to be generalised to problems encountered out of the mathematics classroom. These 'assumptions' and along with them current approaches to using word problems as a teaching and learning tool have been challenged by some authors.

Lave (1992) suggests that approaches to solving word problems seem to be such as to "provide occasions for *separating* math from experience, rather than mathematizing it". Put another way the knowledge, skills and reasoning developed in solving word problems in school may not be appropriate for addressing problems in other aspects of a person's life. The terms 'real life' or 'real world' are often used in relation to word problems. The PNS, for example, discusses the need for learners solving word problems to move from the 'real world' to the 'mathematical world' as this will provide "an uncluttered and generally consistent 'world' within which to work (DCSF 2006a). The implication here is that the word problem is the 'real world'. Verschaffel et al. (2000) explain that a traditional view of working with word problems is seen by many pupils as a 'game', with particular assumptions, that is played in mathematics lessons that doesn't necessarily have any value or application outside the maths classroom. They describe a number of implicit assumptions or rules that may define how the game of word problems is played. Included in these 'rules' are the assumptions that every word problem contains all of the information needed to solve the problem and that objects, people and places that are referred to in problems are different to objects, people and places in the 'real world'. For example the people in word problems seem to be of a standard size, don't seem to suffer from fatigue and don't seem to have the motivations of real people. These and other differences may ensure that the reasoning developed in solving word problems is not appropriate to supposedly similar real world situations.

Gerofsky (1996) also takes this view suggesting the 'claim that word problems are for practising real-life problem solving skills is a weak one" (page 41) citing some similar general reasoning during a detailed analysis of the place of word problems. She presents a discussion in which she attempts to establish word problems as a separate genre with the aim of enabling mathematics educators to see word problems in a new way as "a conceptual object that we will be able to circle around, look at from different perspectives, and compare to other conceptual objects" (page 37). During this discussion she presents an analysis of the structure of word problems and draws attention to aspects of their current use that she sees as problematic. For example, she points out that most word problems seem to be made up of a three-part structure. These components are:

> The 'set up' component, establishing the characters and location of the putative story.
>
> The 'information' component, which gives the information needed to solve the problem.
>
> A question.
>
> (Gerofsky, 1996, page 37)

She explains that there are variations on this basic structure and suggests that advice on solving word problems often relates directly to this structure quoting Johnson (1992) whose advice to students is:

> Look for a question at the end of the problem. This is often a good way to find what you are solving for... What you are trying to find is usually stated in the question at the end of the problem... Simple problems generally have two statements. One statement helps you to set up the unknowns and the other gives you equation information. Translate the problem from words to symbols a piece at a time.
>
> (Johnson, 1992, pages 1 - 2)

Gerofsky argues that advice such as this and the way that word problems tend to be used juxtaposed with practice in relevant areas of mathematics leads to a student's view of a problem solving task that is somewhat impoverished. A task that involves solving a word problem from a student's viewpoint may thus be:

> I am to ignore component 1 and any story elements of this problem, use the math we have just learned to transform components 2 and 3 into correct arithmetic or algebraic form, solve the problem to find the one correct answer, and then check the answer with the correct answer in the back of the book or turn it in for correction by a teacher, who knows the translation and the answer.
>
> (Gerofsky, 1996, page 39)

The reasoning involved in an approach like this does seem very different to approaches that we might take to working on a problem outside the classroom. Gerofsky concludes by stating that it is important not to continue to present learners with word problems that are used in the traditional ways but to consider their 'oddness and contradictions' and develop a rationale for their continued use.

The issue with word problems seems to be less about the problems themselves but more about the ways in they might be used. In the previous subsection suggestions for the effective use of word problems were presented (Mason, 2001; Carpenter and Moser, 1982) but it may also be useful to consider the ways in which the actual use of word problems has been, in Mason's words, "pedagogically abused". In his review of the use of word problems from early times Mason (2001) senses evidence of awareness that as well as working with particular problems, there is the idea that "problems can be stated and solved in general". However he identifies a shift in many authors from seeing (word) problems as a central issue to viewing equation solving as a main focus. In this transition he suggests that the role of word problems in developing an understanding of generality has largely disappeared. Hence he proposes that the potential of the word problem as a pedagogical tool has been at best under-developed and highlights practises that have contributed to this, which he suggests are still prevalent today, including:

- Relegating word problems to the ends of chapters, "so only the quicker students ever encounter them;"
- Using them "as applications of some general technique, and the most important pedagogical aim is to get the students to master the technique, not to resolve (and pose) problems;"
- Presenting problems of a similar type in a scattered way so that the possibility of stating and resolving general classes of problems is obscured;
- Collecting together problems of a similar type "giving the impression of coaching students to solve problems of a given type, but with no further prompt or support for students to extend and vary the 'type' for themselves, and to generalise in different ways;"
- Teachers and students viewing word problems as a burden being the 'hard bit' of maths despite the benefits of the context they provide for working out general techniques;

(Mason, 2001, p. 435)

Word problems still seem to be used extensively in school mathematics albeit in the limited ways that Mason outlines. So in this context do learners respond in ways that are consistent with Gerofsky's argument? Looking at some other authors and studies should throw light on this matter. So what do children actually do when presented with a word problem to solve?

2.6 Children's Responses to Word Problems

Perhaps the most straightforward way of classifying children's responses to mathematical word problems is in terms of whether their answers are correct or not. However such consideration, whether answers are right or wrong, gives no real insight into the strategies used by the child and reveals little about the nature of their engagement with the task. Possibly of more interest, and arguably more useful, is an analysis of the approaches that children have taken in attempting to solve problems. One such analysis is reported on by Larry Sowder (1988) who identifies four broad strategies that children might adopt when attempting to solve word problems. This work will be considered first and will be followed by a discussion of other relevant material.

Coping strategies

The first of Sowder's categories are approaches that he identifies as coping strategies. A child displaying such an approach is likely to scan the text of the problem to locate any numbers that may be present. He then chooses a mathematical operation to perform on these numbers but the rationale for this choice may be independent of the semantic structure of the problem. Thus he may choose the operation that he is most familiar or confident with or perhaps one that he has had a significant amount of recent experience with. This type of approach involves very little real engagement with the problem and it is difficult to see what benefit such activity might provide for the child's wider mathematical development apart from perhaps the effects of some computation practice.

Computation driven strategies

Sowder (1988) suggests that some children might decide on the operation to use by identifying the numbers in the text and using the relationship between them as a possible clue to the required operation. For example, the numbers 78 and 3 might imply that division is appropriate as such a relationship is probably more common for the numbers involved in a division calculation than for other operations. In a similar way the numbers 7 and 8 might imply multiplication. The depth of engagement with the problem that this strategy requires is again limited with the child only using the text base as a source for numbers to operate on. The text base itself plays very little role in helping the child to decide on the appropriate mathematical operation to use.

There is a variation on computational driven strategies in which children may try out each of the four basic operations and choose the one for which the answer seems most reasonable. With this strategy there is perhaps a little more engagement with the text of the problem as the child has to read and understand it to the degree that they can decide what constitutes a reasonable answer.

Slightly less immature strategies

Sowder (1988) identifies a range of strategies in this category. Perhaps the most common involves scanning the text for key words or phrases and using these as clues to the most appropriate operation. For example, "more than" might be seen to imply that addition is appropriate while "less than" might suggest that subtraction is the required operation. It is not uncommon to see posters in classrooms that suggest looking for keywords as well as posters linking particular keywords with each of the four main operations (Butlin, 2003).

Another strategy in this category is deciding whether the answer to the problem should be larger or smaller than the numbers apparent in the text. A larger answer might suggest multiplication or addition while if it is decided that a smaller answer is appropriate then division or subtraction might be implied.

Both strategies seem to require more engagement with the problem than coping or computational strategies. The keyword strategy may require some reading of the text beyond a simple scan and a decision needs to be made about which words are key whereas the 'size of answer' approach requires that the text is understood to such an extent that allows the child to make a decision about the relative (to the numbers in the question) size of answer.

The desired strategy

Strategies in the previous categories may or may not result in the correct answer but none of them are particularly desirable in that they may not involve the child trying to uncover the true meaning of the problem and subsequently choose the mathematics that best fits. This is the approach that Sowder identifies as the 'desired strategy'. This term however does not really give a clear idea of what the 'desired strategy' might 'look like' although the preceding discussion has to some extent indicated what it isn't. It is hoped that this study will throw some light on what might be a 'desired strategy'.

Strategies used early in children's exposure to formal instruction

Carpenter and Moser (1982) in their paper that argues that word problems have a role in developing the concepts of addition and subtraction, give a detailed analysis of the way in which young children respond to addition and subtraction verbal problems. Their interest is in the strategies that children use "prior to and during the first few months of formal instruction in addition and subtraction." They found that children tended to use the same pattern of responses for *Joining* and *Part-Part-Whole* addition problems and indicated that this suggests very little difference in the way that the two types of problem are approached by young children. The initial strategy used tended to be a *Counting-All-With-Models* strategy but over time, with exposure to larger numbers and to situations where no physical aids were available, there was a shift to a more advanced *Counting-On* strategy.

Interestingly Carpenter and Moser (1982) discovered that the children in their study tended to use different strategies for different structures of subtraction problem. In particular the children tended "to model the action or relationship described in the problem rather than attempting to relate the problem to a single operation of subtraction." For example:

- For the separating problem, almost all children used a subtractive strategy (Separating or Counting-Down-From);
- For the Joining (missing addend problem), almost all children used an additive strategy (Adding-On or Counting-Up-From-Given);

(Carpenter and Moser, 1982, p 20)

This relationship between the structure of the subtraction problem and the adopted strategy was observed at the direct-modelling stage but also was noted to continue when children shifted to more abstract counting strategies.

Another, perhaps unremarkable, finding is that for addition and subtraction problems children tended to be more successful with smaller numbers. Carpenter and Moser highlight that a critical stage in children's learning is the transition from informal modelling and counting strategies, as described in their study, to the use of memorised facts and formal algorithms. They go on to cite evidence by Carpenter, Corbitt, Kepner, Lindquist & Reys (1980) which indicates that "by the age of 9, many children mechanically add, subtract, multiply or divide whatever numbers are given in a problem with little regard for the problem's content." It is hypothesised that in the learning of formal procedures children stop analysing problems and that this difficulty can partly be traced to the transition from informal strategies to memorised facts and formal algorithms. The tendency for some children to stop analysing problems is a phenomenon that is reflected in the work of other authors and is further discussed in the next sub-section.

Suspension of sense making

Verschaffel, Greer and de Corte (2000) report on a number of studies of children's responses to what might be called *nonsense* problems. These are word problems that can't be answered using the information in the question. A particular, well known example is:

"There are 26 sheep and 10 goats on a ship. How old is the captain?"

This problem was used by Baruk in a piece of research discussed in the book *How old is the Captain?* (Baruk, 1989). In an article based on this work Selter (1994) reports that of 97 second and third grade French children presented with this problem, 76 children added the numbers coming to the conclusion that the captain was 36 years old. When presented with comparable problems older French children produced similar results.

Many children suggest answers to problems of this sort as a result of carrying out operations on the numbers in the question without demonstrating any real understanding of the text. Pupils' responses to variations on this problem seem to produce high proportions of attempted answers with few children showing any indication that the problems are unsolvable with the given information. With reference to Schoenfeld (1991), Verschaffel et al.(2000) indicate that this type of response to word problems involves a 'suspension of sense-making'. In this case the meaning of the text of the problem is considered very superficially during the problem solving strategy, hence there is little or no attempt by the pupils to make any real sense of the situation described. Interesting but worrying evidence is presented to suggest that there is a link between the proportion of children displaying 'suspension of sense making' and the amount of time they have spent in formal schooling. This is consistent with Carpenter and Moser's (1982) evidence that by the age of 9 many children have little regard for the content of a problem.

"Suspension of sense-making" isn't only evident with *nonsense* problems, it is perhaps just more apparent here than with well-structured problems. The coping and computation driven strategies described by Sowder could equally be considered to involve a 'suspension of sense-making' as they may not involve any real attempt to understand the situation described by the problem.

Making too much sense of the problem.

Cooper and Dunne (2000) considered the role of children's out-of-school knowledge in their responses to assessment items. One particular item involved using a given table, representing the frequencies of different vehicles passing a point, to answer questions about the probability of particular events occurring. For example, children were asked to indicate, by ringing words from a list, the probability of a lorry passing in the next minute. The marking scheme suggested that the answer should be "unlikely", reflecting the relative frequency indicated in the table. However one particular child gave the answer "impossible" justifying his response by using his out-of-school experience of "coming across lorries". In the case of this child his experience seems to over-ride the role of the data given in the question. Cooper and Dunne indicated that over 40% of children answering this particular problem made some reference to their everyday knowledge with just over 20% using everyday knowledge alone. It might be argued that these children were making too much sense of the problem. They had clearly understood the problem situation quite well and had used relevant experience that matched the situation rather than the information presented in the question. Of course this sort of response if given in a mathematics classroom would normally be considered to be incorrect.

"Doing a Paddington"

Strategies such as Sowder's coping and computation driven strategies and those reported on by Verschaffel et al. that involve a "suspension of sense-making" often result in incorrect answers and involve inappropriate engagement with the problem text. However Prestage and Perks (1992) identify an approach that they label "Doing a Paddington" in which the response given by a child to a word problem may be incorrect in terms of what the teacher or examiner was expecting but despite this may have some truth about it and could actually be considered correct if the "implied" assumptions that are normally evident in mathematics questions are dropped. The particular example that Prestage and Perks use to illustrate this phenomenon is taken from one of the Paddington Bear books and involves Paddington answering a question in a quiz. The question involves an 8 foot plank of wood that is cut in half. Each of the two pieces that result are then cut in half and each of these pieces are cut in half again. Paddington is then asked two questions, firstly how many pieces are there and secondly how long is each piece? The expected answers are 8 pieces and 1 foot respectively but Paddington gives 8 feet as the second answer. This is because, in his mind, he has cut the plank lengthways whereas the quiz master cut his plank widthways. Arguably Paddington's answer is quite right. A child who approaches a word problem in this way, "Doing a Paddington", must certainly be engaging deeply with the problem text and forming a good understanding of the situation described. Such engagement would seem to be much more appropriate than that of the coping and computation driven strategies.

The responses considered so far all involve the task of children trying to solve problems. The various categories of behaviour identified above are a classification of the type of activity that children displayed in response to this form of task. There are of course other tasks that can be set with word problems. For example, children might be asked to just identify the operation associated with a particular problem rather than solve the problem. What is less common is a task where children are asked to compare and contrast different problems in some way. However some work has focused on children's classification of problems and there follows a discussion of the responses children gave to a task involving the classification of various multiplication and division problems.

Sorting and matching problems

Lyn English (1997) presented children with two sets each of four source multiplication and division problems. Different sets of problems were based on different underlying structures of multiplication and division. One set comprised of comparative multiplication and division problems while the other involved cartesian product and partitive division problems. Children had to classify the problems in each set according to 'similarity in solution procedure'. The comparison multiplication and division problems proved to be particularly difficult with children focusing on grouping problems according to contextual features rather than the required attention to structure and method of solution. English's study also involved the children participating in other tasks relating to the problems and then attempting the classification task again. Performance in the repeated task was better than in the original sorting task, suggesting the intervening tasks resulted in activity that developed better understanding of the problem situations and structure.

What seems evident from consideration of the strategies discussed above is that many children do not engage in word problems in what Sowder would call the 'desired strategy', but rely on more superficial approaches. It is also interesting to note that in some cases, (like the 'doing a Paddington' approach and the children in Cooper and Dunne's work who used everyday knowledge in preference to information in questions) deeper and arguably more appropriate engagement with the tasks can be rewarded by 'wrong answers' when assessment is made according to the usual assumptions of the mathematics classroom. The 'desired strategy' for solving word problems must be one in which children engage with the problems in depth, striving to understand the particular situations and relationships expressed in them. The issue becomes one of how we encourage children to engage in such a manner. It seems clear that for a significant proportion of children the activity of solving problems may not be sufficient to encourage development of the 'desired strategy'. However English's work involving the classification of problems suggests that there may be tasks, other than just attempting to solve problems, that might result in activity that generates deeper understanding of problem situations.

Consideration of tasks that might help children to develop the 'desired strategy' is a central the focus of this study and is reported on in a later chapter. However a feature of word problems that may have an effect on the way that children work with them is the idea of context. This is explored next.

2.7 Word problems and Context

In the preceding discussion there has been some reference to the *context* provided by a word problem. For example, Verschaffel et al. (2000) suggest that word problems tend to refer to a “an existent or imaginable meaningful context” while the PNS guidance advises that problem-solving “focuses on problems that involve calculations set in wider-ranging contexts...” (DCSF, 2006a). Thus it might be useful to explore just what is meant by *context* and what effects *context* might have on problem-solving activity. So what is *context* in the teaching and learning of mathematics?

Two examples of mathematics in a context

In considering the issue of using and applying mathematics Hughes, Desforges, and Mitchell (2000) in the publication *Numeracy and Beyond* report on the following example from a national mathematics survey:

> ‘It was found that 80% of 12 year-olds could quickly and correctly divide 225 by 15. However, only 40% of the same sample could solve the problem ‘if a gardener had 225 bulbs to place equally in 15 flower beds, how many would be put in each bed?’ Most of the failing pupils did not know which mathematical procedure to use, although they were capable of conducting the routine once the appropriate operation was named.’

(Hughes, Desforges, and Mitchell,2000, page 7)

Marja van den Heuvel-Panhuizen (1999) in explaining some ideas from the Netherlands about the role of context considers the problem of finding the difference in height of two children and comes to the conclusion that '*a lot of children who cannot solve the bare problem (meaning the calculation 145-138) can solve this context problem*'.

In Hughes' example the use of a context causes difficulty for pupils who without it are perfectly capable of carrying out the required operation. However van den Heuvel-Panhuizen is suggesting that the use of context can be helpful to pupils. These examples seem to show that the use of context in the teaching and learning of mathematics is far from straight forward and is an issue that requires very detailed consideration from all professionals who are involved in developing the mathematical understanding and performance of children. However, before the value of using context can be appropriately discussed it may be fruitful to decide exactly what is meant by the term. This is the purpose of the rest of this section.

Common usage of the term 'context'

In the cases above and in common usage in the mathematics education literature the term 'context' tends to be used to imply a description in words (usually written) of a problem situation that may be solved using techniques (often calculations) that are considered to be mathematical. Often these problem situations are said to be 'real life' in that they reflect to some extent a situation that might be experienced in the real world. For example, the PNS

Mathematics Framework (DCSF, 2006a) uses the term 'real life' frequently with reference to the sort of problems that children should solve in the primary school. So, in the case of Hughes' example, the calculation 225÷15 is not considered to be in a ("real life") context, whereas the problem "If a gardener planted 225 bulbs... 15 flower beds ..." is considered to involve a ("real life") context. Boaler (1993) suggests that there prevails a 'belief that mathematics in an 'everyday' context is easier than its abstract equivalent, and that learning mathematics in an everyday 'context' can ensure transfer to the 'everyday' lives of our students'. Such a belief is understandable and was probably held by the authors of the PNS Mathematics Framework. However as the 'flower bed' example demonstrates, not all pupil performance supports this belief. Perhaps by delving more deeply into what constitutes 'context' we may be able to begin to account for this mismatch between belief and performance. An interesting place to start probing more deeply is with questions used in the National Curriculum Assessment Tests.

Contexts in mathematical assessment tests.

Janine Blinko (2004) in considering the role of context in assessment tests in mathematics adopts what seems a wide-ranging definition of context. She begins with a dictionary definition:

> "A definition of 'context' as given in a dictionary incorporates any number of the following words; background, perspective, circumstance, situation, surroundings, environment. A context in a mathematics assessment question also bears such a range of definition."
>
> (Blinko, 2004, p4)

She proceeds by outlining two broad forms of context that are apparent in assessment items from the National Curriculum Key Stage One and Two tests. The first of these is where the context is a mathematical one while the other is what she describes as a 'realistic' context.

Mathematical contexts

Blinko (2004) identifies several types of mathematical context used in test items:

- At one extreme it is argued that a test item like 'Calculate 240 X 7' which might be considered by some to be non-contextual is actually in the context of multiplication being defined as such by the symbols used and the presentation of the item.

- Items that provide a mathematical image such as a number line or a sequence of numbers are considered to use a mathematical model as a context for the question.

- Familiar language is used to embed calculations in items like 'What number is half way between 72 and 184?'

- In some items another area of mathematics is considered to be the context. The use of a Venn diagram to 'sort' numbers according to their factors is used as an example of such items.

This is a departure from the common view of context as many might consider such situations to be context free as in the case of the van den Heuvel-Panhuizen example where 145-138 is considered to be the 'bare problem'. Blinko does of course recognise problems that use the common understanding of the idea of context and in particular identifies 'realistic' contexts.

Realistic contexts

Realistic items are considered to be those that 'model children's possible experiences'. Measurement and money are given as examples of such contexts. However aspects of these questions such as the posing of the question for the learner, the provision of all relevant information and so on is noted as making the questions 'unrealistic' by comparison with the way that mathematics might be used in the real world (Blinko,2004). This recognition that realistic contexts might be analysed further is useful and such analysis has been carried out by Wiliam (1997) in his consideration of the relevance of contexts.

The relevance of context

Wiliam (1997) provides a discussion of realistic contexts that considers the issue of relevance using the metaphor of a MacGuffin. This he describes as a device used, by Alfred Hitchcock, 'to motivate the action in a film, and to which relativity little attention is paid'. The MacGuffin must be seen as being vitally important to the characters in the story but the precise nature of it is of no concern to the filmgoer. For example, 'government plans' was the Macguffin in *North by Northwest* but the exact content of the plans are not revealed to the audience. It only matters that the plans were seen as important to the characters. To the audience the precise nature of the Macguffin is irrelevant.

Wiliam argues that the relevance of the context to some mathematical problems is akin to Hitchcock's Macguffin. The context is there to try to motivate the learner, perhaps to convince them that the mathematics involved has some purpose beyond the mathematics classroom. However in some cases he considers this use to be a '*con'text;* that is the 'context' used serves no other purpose than as 'somewhere for the mathematics to happen'. An example of this he gives is a question that requires determining the fraction of a pint of beer remaining after five-eighths of the pint has been drank. The choice of a pint of beer as the context of the problem is of little relevance as a number of other situations could have been used. Hence the relevance of the context here is like Hitchcock's MacGuffin. There is certainly nothing in the structure of the 'pint of beer' that is particularly related to the mathematical idea of fractions. Of course not all problems are this extreme but many examples of mathematics problems, Wiliam argues, are presented as being of the real world although they have a number of features, that will be highlighted later, that make them stand apart from the everyday situation. These differences may actually cause learners to see less relevance between the mathematics of the classroom and the everyday world rather than bringing the two together as is the common reason for presenting mathematics in a context.

Wiliam identifies another use of context that is based on the relationship between the structure of the context and the structure of the mathematics to be taught. He discusses three aspects of the context that seem to be important in any analysis of contexts used in maths classrooms. The first of these is commonality or the degree of familiarity the learner has with the described context. The second relates to 'match' or the fit between the described situation and the intended mathematics. The final aspect is *range* or how far the context used in a particular task takes the learner. Wiliam gives the example of using temperatures as a metaphor for negative numbers as having a limited range. This context is useful for ordering integers but doesn't lend itself to finding realistic situations for carrying out operations such as multiplication. The aspects of commonality, match and range clearly have an effect on access to the mathematics involved with a particular context metaphor and perhaps could be seen as being a part of the 'greater context'.

A different view of context

So far discussion has focused on context in the sense of what information is supplied with a question or piece of mathematics but there are other ways of looking at the idea of context. The appearance of a question or piece of mathematics might be considered to be part of the context. This interpretation of 'context' has been used by Marilyn Nickson and Sylvia Green (1996) in their investigation of children's responses to 'contextualised' and 'context free' questions (CQs and CFQs). CQs in this study tended to have an 'everyday theme' but the context is also considered in terms of presentation with features such as pictures, words, numbers, symbols and graphics forming the visual impact. The extent to which the presentation of a question draws on these features may determine the 'richness' of the contextualization. At the 'poorer' extreme of this view of contextualisation perhaps all questions might be considered to have some degree of contextualisation as it is difficult to conceive of any way that a question could be presented that doesn't draw at all on these features.

This broadening of the view of what constitutes context is important because the 'realistic' situation that is described in any 'real life' mathematics question is not a true context but some reflection of a real life situation. A further analysis of Hughes' 'flower bed' problem may serve to indicate that often the mathematics textbook or test version of a situation may not be a close reflection of the real life situation in a number of ways.

Just what is the context of the 'flower bed' problem? Many would suggest that it is a gardener planting bulbs, but is it? Another way of looking at the situation might be to consider the context of the problem to be words written on paper. If we take this view then there are some important differences between the written problem and the real situation for a gardener planting bulbs. (Using an analysis in the spirit of an example by Verschaffel et al. (2000)) These might include:

- The motivation for solving the problem. Presumably the gardener in everyday life would be considering the situation because he wanted to or needed to in order to achieve the goal of developing an attractive garden. The child in the classroom may be tackling the problem to satisfy the teacher or to pass a test or exam. The gardener's motivation might be more intrinsic while that of the child might be considered more likely to be extrinsic.

- Accessibility to relevant information. The child in the classroom has all the relevant information presented to him or her in the statement of the problem and in this case there is no irrelevant information. The gardener may have to seek out and find the appropriate data and may have to take into account other factors such as budget, quality of soil and so on.

- The time constraints may be different. The child is likely to have at best just a few minutes to solve such a problem in written form. The gardener may have little time constraint and may consider his options for planting over a period of days or weeks.

- The real context of the problem is different. The child is reading words on paper. The gardener may be standing in the garden looking at the flower beds and the bulbs and thinking about the various options open to him.

- The success criteria are different. For the child the only correct answer is 15 whereas for the gardener there may be a number of equally successful results in the formation of a beautiful garden.

- The written problem is idealistic. All flowerbeds are considered to be the same size and bulbs to be of the same variety. In reality flowerbeds may vary greatly in size while a number of different varieties of bulb may be used by the gardener.

- The strategies to solve the problem may be different. The child will have to read in some detail perhaps using keywords as a clue to the mathematics to use. The calculations for the child may be more likely to be in the form of an appropriate written algorithm taught in school. The gardener may have little reading to do but may be in a position to 'see' and 'feel' the problem. It is likely that his calculation method will be of an informal nature and possibly of a method that he has developed for himself.

Consideration of these differences would suggest that the written form of the problem experienced by the child is certainly a very different context from the 'real' situation for the gardener. The child's context is really 'words and numbers on paper'. This is often the case in mathematics lessons. This would certainly be true of the use of the term context as used in the PNS Mathematics Framework, associated materials and many commercially produced mathematics schemes. In order to analyse context further there is perhaps a need to distinguish between the 'words on paper' context and the context described by those words. For the purposes of this work the former context will be referred to as the actual-context while the later, which is the common usage of context, will be termed the imagined-context.

Actual-contexts and imagined-contexts

An aspect of Blinko's (2004) research involved looking at the effects that the visual impact of questions had on learners. This is looking at the actual-context experienced by the learner rather than imagined-context that is 'described' by the visual features. The features of questions that Nickson and Green (1996) were looking at were also features of the actual-context but actual-context is much more than the visual impact or presentation features of a question.

Where a question (or any other piece of mathematics) is has an effect on the actual-context experienced by the learner. A test paper, with its particular format, might in itself be considered to be a particular actual-context in which mathematics takes place. Textbooks, workbooks, worksheets, whiteboards, computer screens and interactive whiteboards might also be considered as different actual-contexts in which the teaching and learning of mathematics might take place, particularly if the view of context as 'visual impact' is taken, as they each have their typical forms of presentation. It is worth considering some of the features that distinguish between these different actual-contexts.

Textbooks are of a size that is suitable for an individual learner to work with and have a corresponding small font size. Workbooks are often of a slightly larger size in order to accommodate an important feature of their structure, the inclusion of clearly defined spaces in which the learner is required to record their responses to tasks and questions. Worksheets are similar to workbooks but are usually presented as single sheets rather than booklets. Whiteboards are larger and are usually fixed to a wall where the whole class can easily see them although there are smaller portable versions used in many classrooms. Usually the writing and drawing on whiteboards is done by hand, often in the presence of the learners, and can be easily erased. Writing on whiteboards is larger than that in textbooks, workbooks and worksheets and being hand written is of a different appearance. There are perhaps other features that distinguish between these different actual contexts. A similar kind of analysis can be made of computer screens and interactive whiteboards but it is not just in terms of 'visual impact' that these forms of presentation might be considered to be different actual-contexts. The nature of 'activity' promoted by the particular mode of presentation might also be considered part of the actual-context.

A test paper has questions set in particular imagined-contexts. At another level the test paper itself might be considered to be an actual-context but in addition there are 'ways of working' associated with tests that form a part of the test context. Working individually to a time limit with no assistance and in absolute silence is part of the actual-context of a test. Other features of a test context might involve the particular layout of furniture and perhaps the choice of a specific type of room such as a school hall. These features, along with the visual impact features indicated by Blinko form part of what could be called the 'test' context. This context could be compared with the use of an interactive-whiteboard in the mental and oral session of a mathematics lesson. In the latter case the whole class share one prominently positioned IWB and as well as the differences in presentation afforded by the IWB, by comparison with the test paper, there are different ways of working in the 'mental and oral context'. For example discussion is encouraged with a constant dialogue being maintained between teacher and pupils. The children may be sitting on the carpet in a group and have a variety of devices available to use as a means of responding to questions. Clearly these two situations are very different actual-contexts in which mathematics is taking place and of course a wide range of other classroom situations could be viewed to be different actual-contexts in the same way.

The classroom and school as contexts

The situations discussed above all take place in classrooms or other space in schools. At one level of analysis the particular physical organisation of individual classrooms and schools might be viewed as an actual-context. Different classrooms in the same school may be organised in different ways, making each of them a different actual-context in which learning takes place. A particular example of an 'organisational classroom context' is the open-plan classroom which is a very different situation to a traditional classroom which is the home for one class with perhaps the tables and chairs in rows all facing the front. Different schools also have different physical features and organisation that may make them different actual-contexts. For example, some schools have several classrooms organised around a central shared hall while others have long narrow corridors giving access to classrooms. Some schools have buildings set in spacious green fields surrounded by trees while other are entirely set in 'concrete' with no hint of greenery on the site. Schools and classrooms can provide very different 'contexts' in which mathematical and other learning is meant to take place. However it is quite possible for the 'same' physical structure to provide a different context for learning both in time and location.

The school nearest to where I live is a comprehensive built in the 1970s. It was however built as a middle school but never operated as such due to a change in LEA policy during its construction. The first school that I worked in some twelve miles away was a middle school built to exactly the same design as my local secondary school. Middle schools like the one I worked in have a particular way of working, a way of working that is very different to that of my local comprehensive. Despite a very similar physical environment the different ways of working in the two schools resulted in a different actual-context for the pupils in each of the schools. My first school building still stands and is much the same in structure as it was when I worked there but it now houses a 3-11 primary school. Primary schools work in different ways to middle schools anyway but in this case there are differences due to the passage of time. Since I left the building there have been many changes in education that have had an effect on the ways of working in the school. The implementation of the National Curriculum, the NNS (and NLS), national testing, Ofsted, target setting, the PNS and so on will all have had an effect on the context for learning experienced by the children in the school.

Steig Mellin-Olsen (1987) considered 'context to be just about anything that that embodies learning'. This is a view that is consistent with the picture developing from the preceeding discussion but it is perhaps useful to look in a little more detail at the structure of this 'anything'. Jean Lave (1988) provides an approach that goes at least some way to doing this.

Arena and setting a context for each individual

"Activity such as arithmetic problem solving does not take place in a vacuum, but rather in a dialectical relationship with its settings."

(Lave, 1988)

Jean Lave, in arguing this view, provides an analysis that may be useful to employ as a framework through which to probe the mathematics classroom as a 'context'. The analysis considers both cognitive and environmentally determinist views of 'context'. In the case of the latter the tradition of behaviourism, in which 'the context of activity is equated with environment, which determines behaviour' has strong influences. The cognitive view she equates with a 'phenomenological characterisation of "context"'. Here it seems that the focus is on social interaction. However she sees limitations with both approaches describing environmental determinism as a 'system without individual experience' and the phenomenological standpoint as 'experience without system'. She then describes a 'model' of 'context' that to some extent might be seen as uniting the environmentally determinist and cognitive views.

Lave's model is described with reference to a supermarket. The supermarket is considered to be "an 'arena' within which activity takes place". Its form is not normally directly influenced by the individual but individuals may engage in activity within it. The structure of the supermarket clearly must have some effect on the activity of the shoppers and so perhaps to some extent behaviourism may play a part determining what 'context' might be. However to anyone who has ever been in a supermarket it is clear that not all shoppers behave in the same way. So the form of the supermarket as an 'arena' is not the only factor that constitutes the context for the shopper. Shoppers are not identical. They have different needs, motivations and experiences that they bring with them on entering the supermarket. These factors may make individual shoppers view and interact with the supermarket differently. They each have their own individual personal relation with supermarket. This relation Lave describes as a 'setting'. Within any one supermarket there will be as many 'settings' as there are shoppers. Indeed within any one arena there will be as many 'settings' as there are individuals.

Perhaps a useful way of looking at context is to take the view that for any individual the relation that Lave calls 'setting' is in fact the context for that individual. It is this relation that seems to determine behaviour in the supermarket. Change the form of the arena and as any shopper who has visited a supermarket after a 'reorganisation' will tell you the nature of their activity in the supermarket will change. In this case the arena has been altered so affecting the setting resulting in a variation in activity. On the other hand a shopper may enter the supermarket on the way to work with a limited amount of time to spare. Clearly the activity of this shopper will be different to that which they may exhibit when they have no time constraints to shop to. In this case the arena is more or less constant but the setting has been altered by the motivation that the shopper has brought to the activity of shopping. A change in the individual has thus altered the setting and resulted in a change in the nature of their activity.

Clearly changes in setting can come from changes in the arena or from changes in what the individual brings to the arena. However there may be another way in which context may change for an individual. The individual's repeated activity in an arena may in itself alter the setting as they come to learn about the environment they are functioning in. An experienced shopper behaves differently to a novice usually for no other reason then the results of reflecting on experience. Thus experience changes the setting for an individual as it changes what the individual "brings" to the arena. Not only does it seem that there are as many settings as shoppers but the setting of individual shoppers may change over time. If context for the individual can be considered to be the setting, perhaps the context of the supermarket is different for different individuals and different for the same individual at different times. So context may not only be 'anything that embodies learning' but may be different for each individual and may be constantly changing.

The classroom as an arena and setting

There are obvious parallels between the classroom and the supermarket. The physical environment of the classroom with its particular organisation is clearly an arena in which activity takes place. Learners who come to the classroom with their own unique experiences and dispositions have their own individual relation or setting with respect to the classroom. With time and experience of acting in the arena of the classroom it is likely, even desirable, that a learner's setting and subsequent activity will alter. There are ways in which the classroom and supermarket are different. There is no obvious equivalent of the teacher in the supermarket arena. There is a manager and workers who may alter the organisation of the supermarket, put up information posters, even make announcements and answer shopper's questions, but the degree and nature of interaction between the supermarket staff and the shopper is very different to that of the teacher and his or her pupils. A teacher is working in ways that constantly try to create changes in the activity of the learners and so is trying to change the setting from their point of view. The teacher monitors, assesses, intervenes and plans to bring about such changes. However the setting for the teacher also changes over time. Professional development, experience and different classes all may affect the teacher's setting. The control they have over the organisation of the arena is great and of course the changes that they make with the intention of affecting the learners will also affect their own setting.

Clearly the physical organisation of the classroom arena as made by the teacher will have an effect on the individual setting for each learner. However it is perhaps factors that affect the working practices and social structure of the classroom that arguably have a greater effect on the individual's setting. The working practices and social structure in any maths classroom may be influenced by:

- the teacher's beliefs about the nature of mathematics;
- the teacher's beliefs about how children learn mathematics;
- the teacher's predominant teaching style eg: transmission, connectionism etc
- the teacher's feelings about mathematics;
- the teacher's experience as a learner;
- pressures such as inspection, target-setting, performance management etc;
- the time of year, day of week, time of lesson in the day;

Predominantly this list of factors relates to the teacher, but the learners can also influence the setting. For example, the work of Cooper and Dunne (2000) demonstrates the difficulties that some children from some socio-economic groups have in interpreting test items in the ways expected in the working practices of the classroom. Also Zevenbergen (2004) has shown that many of the same groups of children bring to school a language competence that is in some aspects different to the 'middleclass' dialogue that is typical of schools. Thus the relation to the classroom arena of these children is different to that of other groups, hence different settings for different learners.

Wiliam's idea of commonality has a role in the determining the setting for individual learners. Children's familiarity with a particular imagined-context used in a mathematical task may vary greatly throughout a single class thus ensuring that the actual-context is different for each child. Some will find that the imagined-context is familiar and relatively accessible while others may have little experience of the imagined-context and find a particular task virtually impossible as a result.

So what then is context?

> "Context is just about anything that embodies learning"
>
> (Mellin-Olsen, 1987)

If consideration of context is limited to imagined-contexts then there seems to be a mismatch between Mellin-Olsen's views and this common usage of the term. However, if as this work has suggested, the view of context should be broadened to include actual-contexts then it becomes clear that "context is just about anything". This section has attempted to analyse superficially this 'anything' and concludes with a summary of the resulting structure and a very brief discussion of why such an analysis might be important.

Learners function in an actual-context. This actual-context is often the classroom arena. The classroom arena consists of the physical embodiment of the classroom with its particular organisational features. This arena also includes particular working practices and social structures, often strongly influenced by the teacher. The learners bring to this arena their own experiences and dispositions and so each has their own relation with the arena, their own setting. As learners are unique there are likely to be as many settings, and hence individual contexts, as there are learners. The individual settings may (and should in a classroom) change over time as a result of changes in the arena and changes in the individual as a result of experience. Within this actual-context a teacher may use imagined-contexts in the form of printed material or verbal presentation as a means of trying to demonstrate the relevance of the mathematics being learnt and to motivate learners; or as a metaphor that shares some underlying structure with the mathematics being taught. The imagined-context is never likely to be completely consistent with the situation that it reflects and in some cases only has a tenuous link with it. Any particular imagined-context provides varying levels of accessibility to the learners and so also has an effect on the individual's setting. Arguably context is thus peculiar to the individual.

Is considering context important?

Watson (1998) highlights a growing awareness among mathematics educators, over the last twenty years or so, "that the way people learn and do mathematics in school mathematics classrooms is significantly different from the ways they learn and do mathematics in other areas of their lives." The work of Lave (1988) and Nunes, Schlieman and Carraher (1993) among others provide illustrations of this difference. Not only does there seem to be a different way of learning and doing maths in these contexts but there is evidence to suggest that people display difficulty in transferring what they have learnt between these contexts (Nunes et al., 1993; Hughes et al., 2000).

Brown, Collins and Duguid (1989) in discussing *Situated Cognition and the Culture of Learning* provide an account-for these phenomena. They explain that many educational practices tend to

> "assume a separation between knowing and doing, treating knowledge as an integral, self-sufficient substance, theoretically independent of the situations in which it is learned and used."
>
> (Brown, Collins and Duguid, 1989, page 32)

They suggest that the primary concern of schools is often the transfer of the abstract, decontextualised concepts that comprise this substance. The context and ways in which concepts are learnt seems to be viewed as neutral with respect to what is being learnt. Brown, Collins and Duguid explain that this idea of a separation between what is learned and how it is learned and used is now challenged. They write:

> The activity in which knowledge is developed and deployed, it is now argued, is not separable from or

> ancillary to learning and cognition. Nor neutral. Rather it is an integral part of what is learned. Situations might be said to co-produce knowledge through activity. Learning and cognition, it is now possible to argue, are fundamentally situated.
>
> (Brown, Collins and Duguid, 1989, page 32)

They suggest that many teaching practices do not take account of this view of the development of knowledge suggesting that learners are "too often asked to use the tools of a discipline without being able to adopt its culture". For example, they explain that word problems are encoded in a way that is only common to problems of this type and that the practices involved in solving them are foreign to the culture of authentic mathematics practice.

So, if cognition is situated in this way, then educators need to probe deeply into the context of any learning, as experienced by the particular learner, if they are to facilitate the learner using and apply skills and knowledge in 'new' contexts. This means looking at what the learner brings to the classroom as well as at the classroom practices and teaching materials. Educators must also realise that imagined-contexts are very different from the 'real' contexts that they are meant to reflect and so should be careful about the strategies they encourage learners to adopt in tackling them. Some strategies that teachers encourage children to develop to address imagined-contexts are of little or no use in solving a *real world* version of a similar situation. Such a strategy might be the key word approach to solving word problems. For example, in the actual-context of the classroom arena a learner may focus on the key words 'place equally' when reading Hughes' 'flower bed' problem. These words may help the learner decide on division as an appropriate operation to use with the problem. In the arena of the garden the gardener has no access to a written version of the problem he is faced with and so can't possibly use the keyword approach. Hence the keyword approach is a strategy that might help to solve the written form of this problem but provides no help whatsoever for the gardener. This discussion suggests a question that perhaps needs further investigation:

> Is the reason why learners often find transfer, or using and applying, so difficult in mathematics due in some way to the development of particular strategies for solving problems, set in imagined-contexts, that are **only** useful in the particular actual-context in which they are developed?

In his discussion of *Situated Cognition and How to Overcome It* Bereiter (1997) provides an analogy that seems to inform attempts to address this question. He explains a process often used by psychologists where rats are 'taught' to learn a fixed route to a maze under tightly controlled conditions. He points out that if the conditions are changed slightly they are lost but if they are left to run around on their own they quickly learn the whole maze and can find and use efficient routes. Some strategies for teaching word problems, such as the keyword approach, are rather like the tightly controlled conditions that Bereiter's rats experience. These conditions are aimed at getting the rat to produce the required behaviour (reaching the goal) not to 'understand' the maze. Similarly an approach like the keyword strategy is aimed at getting the learner to produce the correct answer rather than increasing their understanding of the mathematical opportunities offered by word problems.

Using the analogy with rats learning a maze perhaps we need learners to "run around" word problems a bit. It is hoped that this study might provide some suggestions about the ways in which we might encourage learners to engage in this 'running around' or exploration of word problems.

2.8 Mental models in problem solving

In her paper that reports on children's classification of problems, English (1997) discusses a model that addresses the potential steps in solving word problems. The model is made up of three parts:

- The problem-text model
- The problem-situation model
- The mathematical model

The problem-text model is the resulting mental representation of the problem text constructed by the child in the process of comprehension of the text. If this model is to be an appropriate mental representation of the problem then the child must identify the relevant 'objects' of the problem; their respective roles and the relationship between them. Of course the goal of the problem needs to be identified.

The problem-situation model is a "mental representation of the situation described by the text" and results from a mapping of the problem-text model "onto an analogous situation whose structural properties are known". This model needs in particular to reflect the relational structure of the 'objects' referred to in the problem text. English considers the appropriate development of the problem-situation model to be crucial to successful problem solving as it forms a link between the problem-text model and the mathematical model. The latter being the result of a mapping of the appropriate mathematical operations onto the problem-situation model. Clearly if there is an inappropriate mapping at this stage then it will be difficult to ensure successful completion of the problem.

Although this seems to be the final stage of English's model this is not the end of the problem solving process. In fact Verschaffel et al. (2000) present a model which outlines additional stages. These include a mathematical analysis of the mathematical model which results in "derivations from the model" which are then interpreted to form the results of the problem solving process which may then be communicated by some form of report.

Both English (1997) and Verschaffel et al. (2000) suggest that when stages of the system are omitted or when models are poorly formed then children will have difficulty in successfully solving a problem. In particular they identify that superficial problem solving strategies (presumably coping, computation driven and keyword strategies would be examples) often omit the problem-situation model. Thus any understanding of the roles and relationships between the elements of the problem will be at best limited.

If we wish to improve children's performance in word problems and ensure that working with them also contributes to their overall mathematical development then it seems that we need to find ways of encouraging them to develop appropriate problem-situation models. It is an aim of this work to investigate tasks that might engage children in such a way as to develop strategies that will help them to construct robust and accurate problem-situation models and hence be able work successfully with problems in a variety of ways. Implied in English's (1997) discussion is the idea that the learners need to engage in a form of analogical or metaphorical reasoning when developing appropriate models of the problems that they are addressing. The next subsection provides an overview of this form of thinking.

2.9 Metaphorical reasoning

Metaphorical reasoning seems to be about making comparisons between concepts. Sfard (1997) citing Pressmeg presents the idea of metaphor:

> As a mapping between two concepts, one of which –the *tenor* – is described (or explained) with the help of the other, the *vehicle*. The basis of the comparison is the *ground* – the set of common features of the tenor and the vehicle.
>
> (Sfard, 1997, page 342)

A particular tenor and vehicle will not share all of each other's features, this would be an isomorphism. The set of characteristics that differ between the components of a comparison are a set referred to as the tension. In this explanation and others (Gentner,1983) there is an implication that the vehicle (sometimes referred to as the base or source) is better understood than the tenor (target). This is emphasised by English (1997) who explains that learners 'need to understand the structure of the source and must be able to recognise the correspondence between source and target". (page 5). In the context of this study, as will be discussed later, this mapping process is not seen as being quite as one-way as implied above. Lakoff and Nunez (1997) discuss two sorts of metaphor. Grounding metaphors are those that relate mathematical ideas to everyday experience whereas linking metaphors relate one area of mathematics to another. In this study learners work on problem statements which could be thought of as indirect experience but it might be that some learners have never experienced the situations described by the problem statements used.

An important point made by Sfard (1997) is that any similarity between tenor and vehicle is created in the learner's mind and is not given to them. This is despite the fact that a task designer may have particular similarities in mind when selecting a particular *vehicle*. An aspect of this study is consideration of this process of creating metaphorical mappings. Gentner(1988) has investigated this process extensively and claims that children "can produce and comprehend metaphors based on common object attributes before metaphors based on common relational structure" (page 48). A central idea of Gentner's work on metaphor is the *relational shift*. This is a change "from primarily attributional to primarily relational interpretations of metaphors" that appears to come about with increasing age. Knowledge of this phenomenon may be of particular value when considering the aspects of word problems that learners are attending to.

2.10 Awareness and attention

Completing tasks of various kinds in mathematics lessons does not seem sufficient in itself to ensure that learners make mathematical sense of what they do (Mason, 2004a). Brousseau (1984) describes an implicit relationship between teachers and learners where teachers set tasks and learners do them. In this *didactic contract* learners expect that if they fulfil their side of the bargain, that is to work on the tasks, then the required learning will take place. However this arrangement develops a *didactic tension* where the more explicit the direction and instructions of the teacher the less likely the learner is to engage in the required behaviour with understanding. As Brousseau puts it:

> Everything he (*the teacher*) does to make the pupil produce the behaviour he expects tends to deprive the pupil of the conditions necessary for understanding and learning the notion concerned. If the teacher says what he wants, he can no longer obtain it.
>
> (Brousseau, 1984, p110)

Mason explains a further part of this paradox in suggesting:

> The less explicit I am about my aims and expectations about the behaviour I wish my pupils to display, the less likely they are to notice what is (or might be) going on, the less likely they are to see the point, to encounter what was intended, or to realise what it was all about.
>
> (Mason, 2004b, page 82)

The general discussion he puts forward suggests that learning can be difficult to bring about and that trying to 'cause' learning can often lead to tears.

In fact it is suggested that learning should not be seen as a relatively simplistic cause-and-effect mechanism (Mason, 2004a). The power of human beings to direct their attention is put forward as a reason for this. However it is pointed out that this control and a human being's power to harness their energies are not always exercised. An important aspect of schooling is seen as providing the "conditions and experiences through which and in which learners discover that they can make choices, control the focus of their attention, harness their energies, and develop personal discipline." (Mason, 2004a, page 2)

Establishing these conditions and experiences seems not to be straight forward. Consideration of the distinction between task and activity (Christiansen and Walther, 1986) might support this claim. Tasks are defined by teachers and authors; they are what learners are given to do. Activity is what learners do in response to a task. What learners actually do in response to a task may often not be what was intended by the teacher or the author of the task and of course different learners may engage in different forms of activity in response to the same task. Mason and Johnston-Wilder (2006) describe that in general terms this activity may be *accepting* or *asserting* in nature. Asserting behaviour involves learners making decisions, forming conjectures, trying things out and generally 'taking risks' in the sense of not worrying about being correct. However it is suggested that in many cases this sort of behaviour does not occur and learners tend to wait to be told exactly what to do (Brown and Kuchemann, 1976). This is accepting behaviour. In such instances it would seem that a learner's attention is not focused appropriately on the task in hand. Mason (2004a) suggests that the important aspect of any task is "what learners are attending to, and how".

This leads to two fundamental questions of education:

- What are learners attending to?

- How can we enable teachers to see what students are attending to?

(Mason,1998, page 247)

Mason (1998) points out that we should not expect a definitive answer to either question but does suggest that the more aware one is of what learners see and what they are stressing and ignoring the more help one can be in trying to direct their attention. If attention is important then it is appropriate to try to explain what it is. However Mason (1998) points out that "it is very hard to define attention, precisely because, in some very essential sense, we are our attention and we are where our attention is" (page 251). He does though describe a dual form view of attention explaining that "part of us gets caught up in doing" while another part works as an 'internal monitor-witness that observes". The inner monitor is a witness to awareness. It is not your attention but co-present with it. The purpose of education being to "awaken and make use of the 'inner watcher' (Mason, 2004b, page 33) thus to develop awareness.

Mason explains that we can be explicitly aware of aspects of the environment or we can be potentially aware. When driving we are potentially aware of what is happening around us but when something in this environment changes we become explicitly aware. In a similar way incidents in a mathematics lesson may provoke our inner-monitor thus altering our awareness making us "more explicitly aware of some features and less aware of others." Awareness however does not work in isolation from other aspects of the human psyche. It functions in a non-hierarchical way, overlapping and intertwining with behaviour and emotion.

> *Behaviour* is what one does observably in the material world. (This can be generalised...) to refer to the manipulation of objects, images, symbols and language.
>
> *Emotion* is the emotional response to a situation, and also the source of energy that drives learners and teachers.
>
> *Awareness* is associated with the intellect; that is, with what one is explicitly and implicitly aware of or sensitised to.

(Mason & Johnston-Wilder, 2006, page 3)

Mason synthesises these components explaining that

> "Teaching is (then) seen as a process of directing students in harnessing of their emotions to provide the energy both to train their behaviour and to educate their awareness."
>
> (Mason, 1998, page 245)

He further points out that when one of these components is pushed into the background a learner's experience is impoverished, and learning becomes impoverished. In fact the view of learning put forward in his writings is closely linked with the idea of awareness. Learning is described as "some sort of transformation in the way that learners perceive or think", the extension of a learner's "sensitivities and awareness" may be such a transformation. Taking this view, a learner 'seeing' or responding to a feature that they have not seen or responded to before would be considered to be learning.

Mason (1998) describes a 'level' of awareness that frequently occurs where we do things without giving them our specific attention. Some of these things are the result of our genetic make up and might include actions that we consider to be innate reflexes. There are other actions that have required learning but once we have reached a certain level of proficiency seem to be done automatically with little explicit attention. He suggests walking, talking, reading and counting as examples. These are behaviours that have involved much conscious attention in learning but subsequently become subordinated in our attention so that this can be focused on 'higher level' goals. For example a driver gets to a state of proficiency where his explicit attention is focused on where he is going rather than on each of the individual actions that contribute to effectively driving a car. This type of awareness Mason refers to as awareness-in-action. He explains that learned awarenesses-in-action develop as a result of natural "powers of construal" which learners have in abundance. He suggests (Mason 2004) attention changes form, often rapidly, "between holistic encompassing (gazing), discerning distinctions (stressing and ignoring, foregrounding and back-grounding), recognising relationships amongst discerned features, perceiving properties that objects in general may possess, and reasoning based on deducing from definitions (selected properties) and axioms" (Mason, 2004, page 9). These various structures of attention are conjectured to be evident in the youngest of children and necessary for learning to take place.

Examples of awareness-in-action are evident in mathematics lessons. For example children recall basic number facts and use them in the performance of more complex calculations. Such behaviour can, and often is, trained without recourse to awareness (Mason 1998). This brings about short term success in routine situations. However this approach may not serve learners well when they encounter novel situations where interpretation may be involved. In other words displaying required behaviour is not sufficient; it is the learner's route to that behaviour that is important. If developing awareness is neglected then learning is impoverished.

Mason (2004) has stressed the importance of shifts in attention. There may be shifts between foci or as described above shifts between forms of attention. For example, attention may shift from gazing at a whole to discerning detail. There appears to be another form of shift, this being one of level. A learner may become explicitly aware of their *awarenesses-in-action*. In becoming explicitly aware learners are able to "distinguish and label both actions and awarenesses, enabling them to study the properties and uses of those labelled actions and awarenesses in action" (Mason, 1998, page 258). It is explained that through this awareness a discipline arises as it is only when we are aware of actions and awareness that they can be studied, formalised into generalities and "assertions of what must, might and may not happen" be developed. For this reason Mason refers to this awareness of *awareness-in-action* as *awareness-in-discipline*. However he explains that when working on mathematics exercises many learners work through tasks rather than on them, thus not recognising the features arising from the tasks that would move them from *awareness-in-action* to *awareness-in-discipline*. The latter is what constitutes the practice of an expert. It would seem to be education's role to provoke a learner's attention in such a way as to develop this *awareness-in-discipline*. Teachers, of course, will have a significant role in this and require a further level of awareness in order to aid them in provoking such a shift.

Mason (1998) citing Dewey (1902) explains this further awareness as being the essence of the difference between the expert and the teacher. Dewey pointed out that an expert is engaged in advancing a subject while a teacher's role is to focus on how "the subject matter can become part of experience" a process called "psychologising the subject matter". The awareness required for this Mason calls *awareness-in-counsel*. Expressed in the style of what has gone before this is awareness of *awareness-in-discipline*. He sees this as a necessary awareness for teachers to call on in provoking learners to become aware of their own *awarenesses-in-action* and hence develop *awareness-in-discipline*. An effective teacher would use their *awareness-in-counsel* to set tasks that would require asserting behaviour and would intervene in ways that may provoke such shifts. This view seems inconsistent with some of the highly prescriptive guidance that is currently presented to primary teachers.

2.11 Motivation

The importance of the *Emotion* aspect of the human psyche has already been introduced in the previous subsection (Mason and Johnston-Wilder, 2006). If a learner is to derive anything from a task then they require an appropriate emotional response to that situation which acts as a source of energy to drive them forwards. Considering this in relation to problem-solving, as Wiliam (1998) points out, "a problem is not a problem if you don't want to solve it". In a similar vein Askew (1989) suggests that "problems do not have an existence independent of problem solvers." This places an onus on teachers to develop and use problems that children want to solve; and tasks that learners want to work on.

Askew (1989), in considering this explains, that problems need to be seen as *necessary* or *curious* by the problem solver. So the motivation for working on a problem might come from a real need to solve a particular problem or it might arise out of interest. An example of the former that I can recall is a situation where I had to change the lock on my front door before going to bed. This was a task I had never done before and was absolutely necessary in the circumstances, hence my motivation. A slightly different, but perhaps equally stressful form of necessity is the need to solve a problem to pass an exam. A problem of the curious sort might be one that I have chosen from the books of Lorraine Mottershead (1978) and Brian Bolt (1984) for no other reason than I found them interesting; or it might be a question like "Why are street lights the height they are?" (Askew, 1989).

Thus teachers need to use problems which their pupils see as necessary or curious. Perhaps a slightly different way of looking at this is that teachers need to work with problems in a way that learners feel is interesting. For example there might be situations that arise out of tasks that involve sorting, matching and constructing word problems that children may find interesting.

Another idea which informs discussion about motivation is Festinger's theory of cognitive dissonance. He presents two basic hypotheses:

> The existence of dissonance, being psychologically uncomfortable, will motivate the person to try to reduce the dissonance and achieve consonance.
>
> When dissonance is present, in addition to trying to reduce it, the person will actively avoid situations and information which would likely increase the dissonance.
>
> (Festinger, 1957, p. 3)

A problem that is unsolved might create a dissonance in a learner that provides the drive for them to engage in activity aimed at solving the problem. Mason (2004) highlights Movshovits-Hadar's view that "Making mathematical findings appear unexpected, or even contra-expected, is the secret of teaching mathematics the surprise way." An example of this is the idea of *torpedoing* (Mason and Johnston-Wilder, 2006). This refers to a teaching strategy that deliberately leads learners to make a conjecture which then has to be modified in the light of further evidence. The further evidence might generate surprise and dissonance which the learner then has to resolve.

On the other hand the second hypothesis might suggest that the dissonance caused by a being presented with a mathematical problem might result in them avoiding problems. This hypothesis might account-for some of the negative attitudes towards mathematics. Thus dissonance seems essential for learning and doing mathematics but may also be a source of aversion towards the subject. It would seem that part of a teacher's role is to manage dissonance in a way that creates drive and motivation rather than ways that might generate fear and avoidance. However a teacher is also subject to dissonance. The choice of tasks and styles of teaching they adopt will be affected by their reaction to the dissonance between the ways of working that they are familiar with and the other ideas they encounter. The implications of such dissonance will be briefly discussed later in the work.

2.12 Summary

This section will summarise the key themes discussed in this literature review and will identify those which will be explored in this particular study.
The essence of a problem is the idea that the route to its solution is not immediately obvious (Haylock , 2006 ; Skemp, 1993). This implies that successful problem solving is likely to involve attempts to uncover the *true* meaning of a problem and choose the mathematics that is most appropriate to use to find a solution. Such attempts would be consistent with Sowder's (1988) "desired strategy".

However there is much evidence in the literature to suggest that a significant proportion of learners do not engage in this way relying on strategies which are independent of the semantic structure of problems (Sowder, 1988; Carpenter, Corbitt, Kepner, Lindquist & Reys 1980). Some authors suggest there seems to be "suspension of sense-making" in children's responses to word problems (Baruk, 1989; Schoenfeld, 1991; Verschaffel at al. 2000). Often these approaches lead to incorrect answers or at best correct answers for the "wrong" reasons. In producing these responses learners often seem to be influenced strongly by particular features of problem statements (numbers, key words) or by their experience outside of the problem itself. For example they might be drawn to using the last operation they practised or be influenced by their real world experience.
The literature seems to suggest that learners' attention is drawn to surface features of problems rather than to embedded structural relationships. Some of the strategies used by teachers do little to shift learner's attention to these

deeper features of the problems. Examples of this are directions to underline key words, Johnson's (1992) advice on working with the three-part structure of problems and the practices identified by Mason (2001) as "pedagogical abuse". The usual approach to using word problems in the classroom appears to be tasks that require children to solve them. These tasks often come after some teaching of an operation and are seen as a means of assessing whether children can apply what they have learnt. The goal is for children to produce the correct answer. However there is discussion in the literature of other ways of using word problems. Gerofsky (1996) states that it is important not to continue to work with them in traditional ways, while Mason (2001) identifies specific examples of non-traditional ways of working with word problems and emphasises approaches that involve educating attention and awareness. This would imply that the role of the teacher is to develop ways of drawing a learner's attention to the structural relationships in word problems. Also of relevance to this study is Moser and Carpenter's proposal that word problems can provide a "viable alternative for developing addition and subtraction concepts in school".

Thus the intention in this study is to explore use of traditional (solving) and non-traditional approaches to working with word problems (sorting, matching and constructing) with a view to:

- Identifying the focus and shifts in focus of learners' attention
- Influencing learners' attention, attempting to draw it towards the embedded structural features of word problems;
- Developing learners' conceptions of addition and subtraction

In doing so it is intended to respond to Gerofsky's (1996) and Mason's (2001) suggestions to move beyond traditional uses of word problems and to throw some light on how we might help children to develop Sowder's "desired strategy".

3.0 The Conduct of the Study

3.0 The Conduct of the Study

My intention in this study is to reveal transitions in children's thinking about, and awareness of, structural relationships embedded in word problems by involving them in tasks focused on working with these problems. In this chapter I will provide detail of the approaches I took to do this, and will identify the nature of the data that I collected. I will outline the approach to analysing that data and provide a rationale for the choices that I have made in shaping the study.

3.1 Outline of the study.

The study discussed in this thesis followed a pilot study that involved a small group of children working on the sorting of problem statements. The pilot highlighted the potential value of sorting tasks as a vehicle for investigating children's thinking as well as developing their understanding of word problems. The main study was designed to probe these areas further and originally involved working with three groups of four children. With one group the focus was on sorting problems, matching problems and constructing problems. One of the other groups worked with a focus on using visualisation with word problems while the remaining group worked directly on solving word problems. Due to the complexity of this and the large quantity of data collected, a decision was taken to report only on the sorting, matching and constructing group.

The main study I report on involved a maximum of eight sessions with a small group of learners at a frequency of approximately once a month. Each session was about an hour long and involved the learners working individually or in pairs on tasks involving word problems. There were four categories of task used with a view to revealing something about children's awareness of the structural relationships in word problems:

- Solving word problems;
- Sorting sets of word problems into subsets;
- Matching word problems;
- Constructing word problems;

In addition to these tasks, in the initial and final probes learners were given number statements to solve as well as problem statements. Two of the learners were involved in additional sessions after the planned final probe in order to provide data that may throw further light on phenomena that had been identified. Flexibility was a deliberate feature of the sessions. This approach arose out of the pilot study where it became evident that a flexible approach which allowed me to react to children's responses was potentially more helpful in revealing their thinking than a rigid pre-planned set of tasks. Thus I had a pre-planned sequence of tasks and some general questions prepared for each session but the precise tasks and questions the learners were exposed to were a result of decisions made 'in the moment' in response to the children's behaviour. A brief outline of the 'content' of the sessions is given below (Table 3.1).

Table 3.1 Outline of Research Sessions

Date	Description of Session
Feb 1	Initial probe administered. Children working alone
Feb 2	Solving, sorting, matching and construction tasks carried out. Children in pairs. Opportunities for paired discussion available.
March 1	Solving, sorting, matching and construction tasks carried out. Children in pairs. Opportunities for paired discussion available.
April	Solving, sorting, matching and construction tasks carried out. Children in pairs. Opportunities for paired discussion available.
June	Solving, sorting, matching and construction (SMC) tasks carried out. Children in pairs. Opportunities for paired discussion available.
July	Final Probe administered
October 1	Session involved Alan and Barry only, working on solving and SMC tasks.
October 2	Session involved Alan and Barry only, working on solving and SMC tasks

Data that I draw on in discussion took several forms:

- Written recordings made by the children. This included answers to problems and number statements as well as 'working out', jottings, constructed problem statements and any other material recorded on paper by the children during the study.

- Transcripts of dialogue. All sessions were recorded on audio tape. Early sessions were transcribed in full but only significant sections of later sessions were processed in this way. However all material on tape was reviewed.

- Field notes. During sessions I made notes of observations that seemed important to record at the time. These notes were supplemented with other material that I recalled from sessions when mentally reconstructing them shortly afterwards.

In addition to this data I have also drawn on my experiences of teaching and learning prior to the study. This has been particularly the case when phenomena have either resonated strongly or conflicted with my experience. Some of the incidents and themes that I have used have been recalled from memory, while others were recorded in brief-but-vivid accounts soon after the event.

3.2 The Subjects and the school

The study took place in a single primary school. An OfSTED Report (OfSTED, 2003) published two years before this study, described the school as follows:

- A large two-form entry primary school;
- Positioned on the border of one of the most affluent and one of the most deprived areas in the borough;
- It represents a culturally diverse population with at least 11 minority ethnic groups other than white British;
- Over 50% of children from minority ethnic groups;
- About 45% of children with English as an additional language;
- About a quarter of pupils eligible for free school meals;
- Just under a quarter of pupils on the register for special educational needs;
- Attainment on entry to the nursery is well below that expected for children of this age;

The report also indicated that standards in mathematics in the school could be improved. At the time of the study the senior management indicated that despite their efforts to encourage a more open and investigational approach to mathematics many teachers in the school, including the class teachers of the children in the study, tended to adopt an approach that might be described as transmission. This is consistent with observations that I have made in the school prior to the study, where lessons seemed to reflect a format of teacher demonstration followed by pupil practice.

There were four children involved in the study, two boys and two girls. They were selected by the class teachers of their Year Four classes who broadly identified the two boys as slightly above average and the two girls as slightly below. According to the school's assessment data the two boys were on track to achieve a Level 4 result in mathematics at the end of Key Stage 2 while the two girls were considered not to be on track for this achievement. I had made a particular request to the class teachers for two children they considered to be broadly above average and two broadly below. Three of the children Barry, Alan and Haleema (*the children are not referred to by their real names*) have English as an additional language while Karen communicates in one language, English. The gender balance and the proportion of EAL pupils were a result of the teachers' selection rather than any requirement of the study.

3.3 Ethical Considerations

The Head-teacher gave her permission for the research to take place in the school. The class teachers in Year 4 were also consulted and after giving their agreement identified a small group of children as possible subjects. In conjunction with the Head teacher I constructed a letter to the parents of these children. The letter explained who I was, gave an outline of the study and how it would impinge on the children's normal school routine. Parents also had the opportunity to consult me. All of the parents gave written permission for their children to take part. Subsequently the selected children were asked if they wished to take part in my sessions. They were told that I was interested in finding out about how children went about solving problems and that by taking part in the sessions they would be helping me. It was made clear to all interested parties that in writing up the study or using the findings in any other way I would not identify the school or the children concerned.

To preserve anonymity the name and location of the school have not been referred to in any material produced by me, and the children have not been referred to by their own names. Tapes of sessions have been kept securely. They have only been reviewed by me and one colleague who helped me with transcription. The colleague was unaware of the identity of the school or the children. The use of any written material that children produced which may have given clues to their identity was restricted to me and my supervisors.

The sessions with me took place in the time periods when the children would normally be in mathematics lessons. The decision to schedule sessions in this way was taken with a view to minimising disruption of their normal school timetable.

3.4 Description of the tasks

Initial and final probes

The same probe was used in the first session and in the final session. It was administered in a test-like situation in which the children were discouraged from discussing the questions between themselves. I provided general reassurance if required but did not provide specific support aimed at helping children to answer the questions. The probe consisted of two parts administered separately in the same session. The first section presented to the children contained seven number statements while the second section contained seven related problem statements. For example the number statement:

$$\square - 6 = 28$$

was considered to be related to the problem statement:

> On Monday morning all the children in Class 2B are present. Six of the children go home ill. There are 28 children left in the class. How many children were present in the morning?

Within each section the order of the number statements and problem statements was randomised so that it was not obvious which number statements and problems were related. The children were given no indication from me of this connection between the number statements and problems, nor were they told that the initial probe and final probes were the same. A copy of the probe items can be found in Appendix 5.

The problem set.

All of the problem statements used in the study were based on addition and subtraction structures. The specific set of problem statements used in the initial and final probes has already been identified. The problems used in the other sessions were drawn from a collection which can be found in Appendix 6. The problems are identified by a coding system. The first three letters of this code refer to the context described by the problem statement. So problems about money contain the letters **Mon** in the code while those about travelling on a bus are indicated by **Bus**. Within each context problems may describe an 'increasing' (**Inc**) or a 'decreasing' (**Dec**) situation. Thus

Fred had £6. His mother gives him another £5 for
his birthday. How much does he have now?

describes a money context and an 'increasing' situation so is coded as **MonInc.** A further variation in problem statements is the 'position' of the unknown. This is indicated by a number in the code. For example, the above problem can be 'modelled' by the number statement:

$$6 + 5 = ?$$

The unknown is the total and is represent by the '?' symbol. This is considered to be in position **1**. If the unknown is the amount that the original total is increased by (the addend) then this is considered to be position **2** while the original amount (the augend) is considered to be in position **3**. Hence the problem statement above is labelled as **MonInc1**. A similar system is used for 'decreasing' situations.

This coding was for ease of comprehension of problem statements by me and by readers of the study and was not made available to the children. When presented to the children problem statements were accompanied by a single letter code that gave no clue about the structure or features of the problem. A copy of the full set of problem statements along with their respective coding and random single letter labels can be found in Appendix 6.

Solving Tasks

Children were given solving tasks outside of the initial and final probes. These usually involved a child being given a problem statement and asked to find an answer in any sensible way. Data about the children's decisions and the reasoning behind them was collected from their recording, observation and discussion.

Sorting Problem statements

Sorting tasks involved children being presented with a small set of problem statements, usually four or six, and asked to sort them into groups in any way that they thought was sensible. Data regarding the result of the sort, the activity engaged in during the sort, and the rationale for a particular result was gleaned from observation and discussion. After an initial sort children were often asked to sort a set of problem statements in a different way. Subsequent sorts in any session sometimes involved presenting a modified version of the original problem set decided in the moment as a response to the children's previous behaviour. This was done with the intention of trying to accustom them to think carefully about the situations they were presented with.

Matching Problem Statements

This task is a variation on sorting tasks. In a matching task a base problem is presented to the children along with a small set of target problems. The children are asked which target problem is most like a particular base problem. There was usually some deliberately chosen variation in feature between the base and target problems. For example the target problems may have a different described context to the base problem. Where matching tasks followed sorting tasks in sessions the matching task often involved the use of some problems that had been used in the sorting task. Data about the children's decisions and the reasoning behind them was collected from observation and discussion.

Constructing Problem statements

In these tasks children were presented with either a base problem statement or a number statement and asked to construct their own problem statement that is like it in some way. Choice of a particular problem statement to use as the base was made during the session and often involved the use of a problem statement that had already been worked on in that session. Data about the children's decisions and the reasoning behind them was collected from their recordings, observation and discussion.

A difference between the sorting, matching and construction (SMC) tasks and the solving tasks is the implication that there is no one specific correct answer expected for an SMC task. It was made clear to the children that I was interested in their ideas and that there was a range of possible correct responses to the tasks.

3.5 Approach to Data Analysis

At the commencement of the study my approach to data analysis was that I would use the tasks with the children and 'watch what happened'. In the initial and final probes some quantitative data was generated and although this was of value in identifying correct and incorrect responses from children, it was less revealing about the children's thinking and awareness than data from the children's recordings, their discussions and from observations of them. In retrospect, analysis seemed to involve several 'sweeps' of the available data. The first stage involved analysis that took place within sessions when incidents were noted and responses were made quickly in the moment. The second stage was analysis that took place between sessions. This usually informed decisions made about the subsequent session. The final stage which was quite lengthy took place following the final session when data from the entire study was reviewed. There follows an attempt to illustrate the nature of these sweeps in more detail.

In- session analysis

Initial analysis took place in sessions when decisions were made quickly during observations of the children and interactions with them. For example if a child seemed to be focusing on the similarity between 'numbers' in a sorting task, I noted this and then presented them with a matching situation in which 'numbers' were not obviously similar. This was with a view to drawing their attention away from 'numbers' and towards some other feature. As decisions such as this one involve the noticing of learner behaviour I consider this to be a stage of my analysis. Responses that I noted in sessions formed part of the data that I drew on in the analysis that took place between sessions.

Analysis between sessions

After sessions field notes were reviewed and additional material added from memory. Tapes of sessions were reviewed. In some cases entire sessions were transcribed while in others although the whole tape was reviewed several times only significant sections were transcribed. Where appropriate, sessions were written up in detail to capture as much of the session as was feasible. Initially this produced an *account-of* the session. Mason (2002) describes an *account-of* as an attempt to describe a situation in a 'dispassionate and objective' way.

However these *accounts-of* perhaps indicate much about me as a teacher and researcher in addition to what they reveal about the mathematical activity of the children. During the first sweeps of analysis the material that I subsequently record as data is a product of the sensitivities that I bring to the study as well as the vast amount of stimuli presented to me in the course of data collection and analysis. As the study progressed I noticed other aspects of the children's behaviour and of the tasks and situations they explored. These subsequent 'noticings' may indicate the development of the learners but they may also provide evidence about my own development. Hence I cannot avoid this study being about me and my development as a teacher and researcher as well as being about the learners.

During construction of *accounts-of* the children's behaviour, experiences of mine from outside the study were brought to mind. These were noted and, where appropriate, have been used as an additional form of data to further illustrate points and strengthen arguments. The *accounts-of* the data were examined in a number of different ways in order to initiate attempts to *account-for* them. Mason (2002) describes *accounts-for* as attempts to offer 'interpretation, explanation, value-judgement, justification, or criticism.'

When reviewing *accounts-of* data between sessions particular themes became apparent. For example at an early stage the tendency for children to sort problems according to the particular keywords or numbers contained in the problem statements was noted. Sorting problems according to the context they described was also evident at an early stage. Having noted features such as these in the data relating to one child I looked for the presence (or otherwise) of such features in the data relating to other children. By this approach I noticed several themes emerging from the data as the study progressed. This analysis also informed the conduct of each session. For example if a child seemed to be drawn to context as a feature to sort by, in the subsequent session I presented them with a small set of problem statements all describing the same general context. This was in an attempt to draw attention to less obvious features. My interest was in drawing children's attention beyond the surface of problems thus as the study progressed the tendency was to attempt to look more deeply at a small range of contexts.

The *accounting-for* which took place between sessions was recorded and was part of the focus for analysis that took place after the conclusion of the final session.

Post session Analysis

It would be cumbersome to attempt to provide a complete picture of how I analysed the data at this stage and how I attempted to *account-for* what I saw but I will give a flavour of my approach. As already implied, analysis begins, perhaps unconsciously, from the first instance of a child observed responding to a task. The observation itself triggers thoughts and awarenesses arising from my previous experience. In my mind the observation may be positioned in relation to these thoughts, perhaps being consistent or dissonant with them. Thus analysis has started. The act of transcribing tapes may also be considered as a stage of analysis. This process as well as converting the sounds on tape into words on paper for easier access caused me to consider very carefully every word that I transcribed. The tapes from a couple of sessions were transcribed for me by a colleague. This was necessary due to demands on time but I do acknowledge that in the case of these sessions I may have lost the opportunities afforded by this layer of analysis. Not all sessions were completely transcribed. In some cases tapes were reviewed several times but only sections considered to be significant were transcribed initially. This selection process was part of my analysis. Sometimes consideration of data caused me to notice features that were not to the fore of my mind when first listening to a tape. In these cases I reviewed tapes in case my new found sensitivities caused me to be drawn towards some feature that I had ignored on an earlier review.

Once *accounts-of* the sessions were constructed these were reviewed and reorganised in several ways. It is worth mentioning at this point that I consider the *accounts-of* to be subject to the possibility of constant development. There was never a point in the study where I ever considered that I had recorded everything available to me in an *account-of*. They were added to in the course of the analysis as different features seemed to emerge and become important to note. The *accounts-of* sessions I constructed are not a complete picture of what went on nor can they ever be. Some potentially useful data was not collected, for example video was not used in the main study, so the capturing of facial expression is not evident in my *accounts-of* except where I have made a record in my field notes. (Videoing of sessions in the pilot was trialled but was rejected as an approach in the main study due to the constraints of time and space imposed by my allowed access to the school.)

The reorganisations were carried out in order to create possibilities for noticing that may not have been evident from my original *accounts-of*. Initially a record of each individual session was constructed using the different sources of data available to me. These accounts were read and re-read looking for themes and significant incidents. I recollect 'seeing' some interesting aspects of the children's behaviour but I do recall 'hitting a wall' in the sense of getting to a point where I was sure that there was more for me to see but I couldn't see it. The first reorganisation of this initial record of the data was the production of Initial/Final probe analysis sheets (see Appendix 7). These facilitated the noticing of similarities and differences in the children's performance in the two probes and subsequently provided a wealth of material to illustrate the main themes arising out of the study.

The next reorganisation was to attempt to tell the story of each child's behaviour over the course of the study. For example, in the case of Karen this involved selecting every reference to Karen in the session accounts and placing them in chronological order in a new account. This account I think of as the 'Story of Karen' as it affords access to changes (or otherwise) in Karen's behaviour over time. A story of each of the children was constructed. As will become apparent this rearrangement of the data was helpful in drawing attention to phenomena such as *persistent behaviour* and *folding back*. This reorganisation was also helpful in drawing attention to changes in a child's behaviour over time. A further reorganisation afforded useful insights into the effects of particular tasks. Initially this manipulation involved collecting together all references to solving tasks in my *accounts-of* and considering them in relative isolation from the other tasks. I then collected all references to sorting, matching and constructing tasks and considered each collection separately. This again afforded the possibility of seeing things in the data that I may not have been drawn to in the other forms of account. The next two chapters discuss what was revealed by considering the data in each of these ways. However first it would be useful to be clear about my focus in this analysis.

In the presentation of any task the role of the teacher or the researcher is obviously very important. There will be, by necessity, interactions between researcher and subjects. In this study instructions were given and children's responses were commented on. It is acknowledged that the particular phases I used, the tone of voice that I adopted and the body language and facial expression that I displayed would all have some effect on the subjects. It is also acknowledged that the timing of such interactions is also likely to have an effect. However my interest was focused on exploring the potential of sorting, matching and constructing tasks themselves rather than on my 'teaching'. Thus I took several steps to minimise the effects that my interactions would have. I aimed to provide brief instructions and as similar structured tasks were repeated from session to session I tried to use similar language on each occasion. I consciously did not intervene in a task until it was clear that the learner had completed the task or had come as close as they could to completing it. My interventions involved asking the children to explain why they had produced a particular response. To minimize my effect on the interactions I often asked the children to explain their thinking to each other. In my analysis I have chosen not to focus on the particular phrasing, body language and facial expression that I used, which I tried to keep as similar as possible on all occasions. However decisions that I made in the moment in sessions, such as the choice of which problem statement to use as the base problem in a constructing task, are interventions that are considered in the data analysis.

In trying to remove my influence from the conduct and analysis of the study I hoped to be able to make valid and reliable comment about the role of sorting, matching and constructing tasks themselves rather than about the nature of my 'teaching'. I acknowledge that a different decision could have been made which might have involved focusing in more detail on my role. I feel that this would have shifted the emphasis of the study more to exploring the potential of my teaching rather than my main interest, that of exploring the potential of sorting, matching and construction tasks

3.6 Rationale for the research approach.

The approach that I have taken to the conduct of this study is broadly in the spirit of 'grounded theory' (Strauss and Corbin, 1998). Consistent with Denscombe's (2003) description of the grounded theory approach, my concern has not been with testing theories but with generating them from data that has been collected. Although bringing my own sensitivities and attunements to the enquiry I tried to embark on the study with the open mind that is necessary for useful theory to emerge. An open mind isn't a blank sheet, rather one that is informed about an area but tries to avoid a preordained way of seeing that area.

The grounded theory approach interweaves data collection with analysis (Denscombe, 2003). It is not a case of collecting a variety of data and seeing what comes from it but a rather more continuous spiral-like activity that involves collecting data, identifying emerging phenomena and theory; and then further re-examining the data (or collecting more) on the basis of what has been noticed in order to further develop understanding of the phenomena and to theorise about it. In the light of this it should be made clear that the approaches to working with the data were not pre-planned at the commencement of the study, but were seemingly useful ways forward that occurred to me in the course of the research. Similarly, as already suggested the format of individual sessions was flexible being dependant on the particular behaviour the tasks elicited in the children, and on my sensitivities at the time. This could be considered to be a weakness, in the sense that the approach has put me in the position of making claims as a result of contrasting and comparing data arising out of dissimilar situations. It is also possible that a different researcher with different sensitivities would be drawn to different aspects of behaviour and make different decisions. This weakness is acknowledged. However I feel that a benefit to the flexible approach that I took is the possibility of probing a child's thinking in more depth than a more rigid approach might afford.

Ginsburg (1981) provides some support for a flexible approach in discussing the 'clinical interview' as an effective means of gaining access to the nature of a learner's thought, which was a main focus of my study. He attributes the development of this technique to Piaget describing it as

> "a flexible method of questioning intended to explore the richness of children's thought, to capture its fundamental activities, and to establish the child's cognitive competence".
>
> (Ginsburg, 1981, p 4)

He argues that the clinical interview, though 'far from foolproof' is the most appropriate method for accomplishing the three basic aims of research into mathematical thinking which he outlines as:

- The discovery of cognitive processes;
- The identification of cognitive processes;
- The evaluation of competence;

Whilst not reflecting every characteristic of the clinical interview my approach does include, along with other techniques, the idea of flexibility in the selection of probes and questions.

An aspect of the clinical interview that I chose not to adhere to on all occasions was a one-to-one relationship between interviewer and child. Sometimes I would encourage the pair of children to talk together about their actions and thought. This was an attempt to minimise the number of interventions that I made which may have affected the children's responses to tasks. This is in the vein of Foster (1996) who favoured such pairing as it minimised prompting from the interviewer and enabled them to focus more directly on observation and data collection.

I have already introduced discussion of *noticing* and *awareness* in the literature review, as prior to the study this area seemed to have relevance to the behaviour of the children with whom I worked. Noticing and specifically *The Discipline of Noticing* is central to the conduct of the research. Whatever I conclude and recommend arising out of this study I will do so as a result of what I have noticed. Mason (2009) explains that 'noticing lies at the heart of any and all research' and so working on noticing can improve research. In fact he goes further to suggest the *Discipline of Noticing* as a research paradigm in its own right (Mason, 2002) arguing that it forms a "well-founded method of coming to know and of validating that knowing in a reliable, robust and generative manner". He suggests that like every other approach to research it does not ensure certainty but that it does provide "a self-consistent way of working, through which the practitioner can take responsibility for remaining in the question rather than committing themself to a single interpretation".

Mason (2009) points out that there are many things that we do not notice. This he attributes to a lack of sensitivity to particular aspects or to our attention being directed to some other feature of a situation. As a means of developing attention and of avoiding repetitive habitual responses to similar situations Mason suggests the *Discipline of Noticing*. Central to this is the idea of 're-constructing' one's experience. In one form this might be carried out entirely in the mind, perhaps in the way that I would think through incidents in lessons whilst driving home after a day's teaching. In another form it might be describing to a colleague "what you saw that others may have seen and can recognise". In such a description Mason stresses that the focus should be on "the behavioural rather than affective or emotive aspects". Yet another form is a brief-but-vivid written *account-of* a particular experience. The idea of this is to present a description of a situation in such a way that the reader (or the writer when reviewing it at a later day) is able to 'enter the situation' and see it from the writer's perspective. This form of reconstruction appears to have several purposes. Written *accounts-of* may provide a useful record of experience that can be reviewed and used by the writer allowing them to consider other ways of acting should a similar situation occur in the future. Even if not recorded in written form the process of mentally reconstructing events may help recall of the incident in the future. By eliminating (as far as is possible) affective, emotive aspects and judgments from an *account-of* provides an opportunity for colleagues to agree about the nature of an incident before moving to a more evaluative stage. In essence an *account-of* is about describing *what* happened rather than explaining *why* it happened.

The material that I have collected and presented as data from this study is essentially a collection of *accounts-of* my experiences during the research. In constructing the accounts my direct experience has been supplemented with consideration of taped discussion and children's recording. From this collection my attention has been drawn to particular phenomena.

> "A phenomenon is something distinguished from the flow of events and recognised as having happened previously; recognising it involves discerning some feature or aspect as (relatively) invariant in the midst of other things changing."
>
> (Mason, 2004, p 2)

I constantly worked on my data to try to discern distinctions that may subsequently acquire the status of phenomena. Having noticed something I then looked for other examples of it in other parts of the data or even collected more data. This was in an attempt to better describe the actual phenomenon that I encountered. When engaged in this activity I noted possible *accounts-for* but then tried to suppress them in my mind in order to concentrate on building a picture of the particular phenomenon in question. This phase of my analysis has parallels with the phenomenological approach to social research (Denscombe, 2003) which seeks to develop a description of the phenomenon itself rather than trying to investigate its causes. However this study goes beyond this level of analysis and attempts to provide explanations of observed phenomena.

I have attempted to present my collections of accounts in such a way as to make the phenomena that I have noticed accessible to others in order to gain consensus about what I have seen. Mason (2009) suggests that having 'agreed' on what has been seen the next stage is to *account-for*. This stage is about explaining why incidents occurred and what has caused phenomena to develop. This stage might include reference to other research, theory and other aspects of experience. It is this stage that I consider to be the main analysis of my data. Mason suggests that some research papers tend to inter-mingle *account-of* and *account-for* and fail to provide the detail required for a reader to reproduce the conditions of the study. This is a 'trap' that I have tried to avoid. However I have presented no chapter or section which is solely a presentation of data alone. To do so in a form that is relatively concise and comprehensible to a reader would be extremely difficult. Hence I have chosen to develop chapters that present data and discuss it. Within these chapters I have attempted to distinguish between *account-of* and *account-for* so that the reader may 'view' particular incidents to better be informed about my *accounts-for* them. I have endeavoured to provide enough information in this form for a reader to follow my reasoning and even for them to construct alternative *accounts-for* the situations described. What I have not done is to provide the reader with access to all of my data in a way that would allow them to carry out their own complete analysis. However an example of my data record is available in Appendix 8 in order to give the reader a flavour of material that I have collected.

My attempts to explain or *account-for* the phenomena that I have noticed is in the spirit of phenomenography. This is an approach to qualitative research developed by Ference Marton which "aims at description, analysis and understanding of experiences" (Marton, 1981). In particular a goal of researchers involved in phenomenographic study is to come to understand the qualitatively different ways that people might think about a phenomenon. In my *accounting-for* my aim was to try to "...take the part of the learner, see the experience through the learner's eyes..." (Marton and Booth, 1997) with a view to coming to understand why the learner acted in the way that they did.

I am aware that I may have given the impression that the transition from *account-of* to *account-for* is a one way process. This has not been the case in this study. *Accounts-for* frequently formed in my mind before *accounts-of* had been confirmed, though in completing the *account-of* I did try to suppress the tendency to include these thoughts. In the processes of *accounting-for* particular incidents and observations I had cause to look at data in different ways and thus created new *accounts-of* or developed existing ones further. It should also be made clear that there may be many possible *accounts-for* any particular observation and that sometimes these *accounts-for* may be contradictory. Where appropriate I have provided alternative *accounts-for* but I have also attempted to make a strong case for those *accounts-for* which seem to me to be most likely.

If it were possible to give others access to all my data and to my direct experience they may draw different conclusions and make different recommendations on the basis of what they notice. I see the writing up of the study as an opportunity, not only to tell others what I have seen, but also as a means of assisting them to see these things as well, and perhaps also creating an opportunity for them to provide other interpretations and explanations. I do not intend to say that what I have found is "how it is". Rather in the spirit of Mason I am trying to present an *account-of* what I have noticed and what I am now aware of. This may be no more or less an accurate picture of 'how things are' than the views of another. What I write though is valid in the sense that it reflects my experience and the work that I have carried out in considering particular incidents from it. What I present, conclude and recommend I offer as a representation of my awareness and I offer it in the hope that it may be useful to others just as the reflections of others have been useful to me.

There follows in the next two chapters the presentation of *accounts-of* my data along with discussion that is my attempt to *account-for* it.

4.0 Children working with solving, sorting, matching and construction tasks

4.0 Children working with solving, sorting, matching and construction tasks

The focus of this chapter is to present and analyse data collected from working with children on solving, sorting, matching and construction tasks involving word problems. I begin by separately discussing each child's response to each of the four types of task used in the study. I go on to identify and discuss themes evident in the data from each type of task and conclude by identifying themes that seem common across tasks. These are discussed in the light of literature already presented; however new material is introduced to account-for observations where appropriate. Where it is considered useful for the reader to see data and the initial *accounting-for* that took place in or immediately after data collection this is presented in text boxes to distinguish it from the subsequent in-depth analysis that took place after all of the data had been collected.

4.1 Discussion of data from word problem 'solving' tasks

In each of the sessions the children were presented with at least one word problem to work on alone with the aim of solving it. Discussion with other children and with me sometimes took place after the initial attempt to solve a problem. The data collected from the work on these problem solving tasks included written answers and other recording, and transcripts of discussion. The latter though was not collected in all sessions. The available data from the problem solving tasks was examined and common themes identified. A discussion of the data relating to each child in turn is presented, followed by a general discussion of the themes that emerged. In this analysis problem-statements are identified by the single letter code used with the children and with my coding system. Where appropriate the *change*, *combine* and *compare* classification (Riley, Greeno and Heller (1983)) of word problems is referred to. (Chapter 3 provides a further explanation of these systems.)

So what happened when the learners in the study were presented with 'addition' and 'subtraction' word problems to solve?

4.1.1 Karen

Karen was perhaps the least confident member of the group and performed least well in terms of the number of correct answers she produced. Only four of the fifteen problems that she was presented with in the study were answered correctly.

In the first session Karen's response to each problem was either to add the numbers in the problem-statement or to indicate that she could not do it. Where an answer was given to a problem Karen produced some form of recording of her working. This was always addition, even for problems where some form of subtraction was more appropriate. The one correct answer that Karen gave was in response to one of the potentially harder problems (Appendix 7: Problem D in February probe for Karen). Could it be that this correct answer occurred as a result of chance rather than by use of a strategy that involved an understanding of the problem-statement? If Karen had some disposition to always respond to word problems by adding the numbers present in the problem-statement then adding would, on some occasions, be appropriate. In such cases, if the addition operation was carried out accurately, then a correct answer would result. Reviewing this first session in the light of Karen's subsequent behaviour strongly suggests that she did have a tendency to respond in this way and so produced the correct answer to this problem without recourse to understanding the problem statement.

Karen was presented with a single problem to solve in each of the February, March and April sessions. On each occasion she appeared to respond by adding the numbers in each of the problem-statements despite this operation being inappropriate. Attempts to discuss Karen's thinking tended to elicit descriptions of how she added the numbers rather than why she used addition. The March session included dialogue typical of this.

Figure 4.1. 1 March Session Solving Task Karen

Problem K (MonInc3)

Sue had some money. On her birthday she was given £16. Now she has £31. How much did she have before her birthday?

JB	Right Karen, what have you done with K?
K	I've done, I've added up the 10 and I've added up the units so now 10 add 30 is 40 and I know that 6 add 1 is 7 so I thought the answer must be £47.
JB	OK, so how did you know you had to add?
K	Because I know that... because I know that 10 add 30 is 40, and I know that 6 add 1 is 7, so that must be what the answer is.

One possible *account-for* this is that Karen's attention is fully required to carry out the operation of addition so she launches into explaining the strategy as, at this time, it is the sole focus of her attention. Karen's facility with addition seems not to be subordinate (Hewitt, 1996; Mason, 2002) to the challenge of problem solving. She appears to have to focus on thinking so carefully on the *how* of adding that she has no remaining capacity to focus on *why* she carried out that particular operation. Development of Karen's facility with addition to the extent that it becomes more automated might enable her to deploy a greater proportion of her attention to the question of *why* she was responding to the problem in this way.

Prior to the start of the final session Karen's responses to word problems could be grouped into one of two categories. These were either responses which involved adding the numbers in the problem-statement (7 problems) or responses which suggested that she could not do the problem (2 problems). The high frequency of use of one particular operation might suggest that Karen was adopting a 'default trigger' response where some feature of the problem-statement caused her to carry out addition without consideration of the problem-statement as a whole. It may be reasonable to assume that Karen's attempts at developing an arithmetic model of a problem involved reading the problem, identifying the numbers present and adding them. However there are two problems (Figure 4.1.2) that she indicated that she couldn't do.

Figure 4.1.2 February Probe Solving Task Karen	
Problem C (Change 6) On Monday morning all the children in Class 2B are present. Six of the children go home ill. There are 28 children left in the class. How many children were present in the morning?	**Problem F** (Combine 2) There are 16 boys in a class of 31 children. How many girls are there?

Why she would deviate from the hypothesised 'trigger' strategy with these problems is not entirely clear. One of the problems (Problem C) was the only problem that expressed a number in words rather than in numerals. The lack of an obvious number in the text may have in some way interfered with the 'default trigger' response. As the number didn't appear in numeral form she perhaps failed to see it. If her attention tended to be drawn to the numerals in problem-statements then a number expressed in words may not be noticed and so would not provide her with the information that she needs to produce an answer. The lack of a numerical answer being provided for this problem could be considered as evidence that Karen was reliant on using a 'default trigger' response to the presence of numerals rather than trying to make sense of the problem statement. The other problem that she suggested that she couldn't do was Problem F.

Figure 4.1.3 February (First Session) Probe Solving Task Karen	
Problem F (Combine 2)	**Problem E** (Combine 2)
There are 16 boys in a class of 31 children. How many girls are there?	There are 32 children in Class 3C. There are 15 boys in the class. How many girls are there?
"Can't not do it"	47 30 + 10=40 5 + 2 = 7

This problem was similar in structure to Problem E (see Figure 4.1.3) and involved numbers of a similar magnitude so it is perhaps surprising that she didn't respond to it in a similar way. In the absence of any other data, accounting-for the different responses should focus on differences between the presentations of the problem-statements. Features that may possibly have influenced Karen are:

- the difference in the order of mention of variables in the problem-statements;
- the presence of the class 'number' in Problem E;
- one problem-statement being expressed in two sentences while the other is expressed in three;
- the place of the particular problems in the sequence presented in the exercise. (Problem E came before problem F, with several other problems in between);

Taking account of some of these features in the design of subsequent problem-solving tasks could assist a researcher in uncovering Karen's reasoning and may also help a teacher in the development of Karen's problem-solving skills.

In the final session it seems from her answers that Karen uses some form of subtraction in response to three of the six problems (Appendix 7: July Probe for Karen). In two of these problems subtraction is an appropriate operation and correct responses are produced. However the problem where subtraction was inappropriately used (Problem D) may tell us a little more about the activity she has engaged in and where her attention has been focused. Up until this session there has been little or no indication from her answers, her working or discussion that she has focused on anything other than looking at the numbers in a problem-statement. A possible account-for her response to Problem D in the final probe is that she attended to the phrase '5 children had gone home' and this suggested 'takeaway' as a suitable operation. Previously no comment, recording or behaviour suggested that attention was given to the words in the problem-statement. In the final session Karen also wrote the majority of her answers in a sentence. These sentences at least referred to names expressed in the problem-statements but in some cases made greater reference to the problem. For example:

"There are 22 people where present at beginning"

"There are 15 girls in the class."

This might suggest that Karen had shifted her attention from looking at the numerals in the problem-statement to looking at the words. Assuming that such a shift has taken place has it just occurred spontaneously or has some experience caused Karen to make this shift in her attention? This is a question that will be addressed in a later section.

4.1.2 Haleema

Haleema's responses seem not to be as dominated by addition as did those of Karen. In the first session she appropriately uses addition and subtraction in answering four problems correctly. She produces no numerical response to the two 'change 6' problems. The overall pattern of her responses suggest that her behaviour when attempting to solve problems is allowing her to make more sense of the problems then did Karen's behaviour at this point. However as no interview took place in this session and as Haleema did a very limited amount of recording nothing more can be said about the nature of her activity in this session.

In the March session she appropriately used addition when working on a 'change 1' problem (Figure 4.1.4 Problem A) and suggested in interview that the words 'got on' in the problem-statement influenced her choice of operation.

Figure 4.1.4 March Session Solving Task Haleema
Problem A BusInc1 There were 27 passengers on a bus. At the bus stop another 14 passengers got on. How many passengers were on the bus now?
Haleema gave the correct answer of 41 and although she showed no working out wrote her answer in a sentence: 'altogether thre are 41 Pasengrs.' She found it difficult to indicate what was 'in her mind' when working on the problem but after a little discussion indicated that the words 'got on' in the question suggested to her that the appropriate operation to use was addition

In the April session she incorrectly answers a 'change 3' problem. Discussion with her doesn't make it clear why she answered in the way she did. In the fourth session Haleema adds inappropriately when attempting to solve problem C. This is a problem that could be represented by the number-statement 35 + ?= 49 and so could be solved by subtracting 35 from 49. She suggests that she added because the problem says that 'people got on the bus' (Figure 4.1.5)

Figure 4.1.5 April Session Solving Task Haleema
Problem C **On a bus there were 35 people. At the bus stop some passengers got on. There are now 49 passengers on the bus. How many people got on the bus at the bus stop?** Karen worked on this problem with Haleema who tended to dominate the discussion. Karen may have been influenced strongly by Haleema's responses. The answer given independently by each girl was 84. Haleema suggested that she had worked this out by counting on and she had counted on because 'people got on the bus'. She followed this by reading part of the problem "There are now 49... "and saying " that's adding on..... I don't know". Karen agreed, at which point Haleema re-read the problem aloud with a particular emphasis on the word 'now' in the phrase "There are NOW 49". Haleema then set about doing another calculation. She again added and got the same answer. Karen followed suit also getting 84. The girls are requested to read the problem again and are asked if 84 is a sensible answer. This is met by a period of silence. The girls are then asked what the 84 refers to. Haleema says that it is a number. Karen qualifies this by saying 'passengers'. When asked what are you trying to find out. Haleema suggests that it is the answer to 35 add 49.

In the final probe session although a number of answers are incorrect Haleema generally uses appropriate operations when working to solve problems.

Haleema seems to be reading the words of problem-statements but her attention is drawn to particular phrases as clues to what to do rather than considering the meaning of the whole problem-statement. This might be thought of as an extension of the 'keyword' approach. Here a phrase is used as a clue to the required operation rather than a single keyword. It would seem useful to Haleema's development if some task could be devised which might help her to focus more on the meaning of the whole problem-statement and build her mathematical model of the problem from this.

4.1.3 Alan

Alan correctly solved a very high proportion of the word problems presented to him. Consideration of the strategies that he used seemed to reveal interesting distinctions in the way that he responded to problems with slightly different mathematical structures. Problem U presented in the second session provided the first opportunity for Alan to talk about his approach to solving a problem (Figure 4.1.6).

Figure 4.1.6 February Second Session Solving Task Alan	
Problem U **Ann's parents gave her some money for her birthday. She already had £25. Now she has £41. How much money was she given for her birthday?**	
JB	Can you tell me what you got for your answer?
A	I got sixteen pounds.
JB	Okay, sixteen pounds. Why is it sixteen pounds?
A	Because I added 20 onto 25 and that was too much and then I went into the teens and I got 16.
JB	Why were you trying to add on to the 25?
A	Because Ann gives... no Ann's parents gave her some money. She has 25 pounds ... (pause) ... because Ann has got 25 pounds and she has been given 41 no... she has been given some money to make 41.
JB	So you were guessing how much she was given and it was too much... so you guessed 20 that was too much so you tried a little less.
A	First I did 20. I knew that was too much, then I went 25 add 15 and that was 40, so I said 16.

Problem U is classified by Riley et al. (1983) as a Change 3 problem. Change problems involve an increase (referred to as the addend) or decrease in an initial quantity (the augend) as a result of some event. In the case of this problem there is an increase in the quantity of money that Ann has as a result of the event of receiving money as a birthday present. This amount (addend) is the unknown that the learner needs to find. It may be useful to consider how this particular situation might be represented with mathematical symbols. Askew (2003) describes the Realistic Mathematics Education process of horizontal mathematizing as 'stripping the essence of the problem out of the context' that leads to a 'mathematical reading' of the problem. In this case that 'reading' might be 'what do I have to add to 25 to make 41'. Expressed as a mathematical model this could be:

$$25 + ? = 41$$

Change problems that involve an increase reflect Haylock's (2006) augmentation structure of addition. This type of situation implies an order and a type of action that may not be suggested by situations reflecting an aggregation structure. In an aggregation or combining structure two distinct quantities are being merged or joined. The description of such a situation does not imply an 'initial' and a 'change' quantity so a learner may not be strongly influenced by the problem in ordering the quantities in their calculation. Hence they might use the commutative property to rearrange the calculation to make it easier and/or more efficient (for example 3 + 98 may be rearranged as 98 + 3). In an augmentation situation the augend and addend are implied by the problem-statement and so may influence the learner to adopt a strategy that involves adding the identified addend onto the initial quantity. This influence might direct attention away from the potential use of the commutative property and as a result also obscure the use of the relationship with subtraction. If subtraction is seen as 'undoing' addition then with the influences that this problem exerts it might be difficult for a learner to see how subtraction would help as the number that needs to be subtracted to 'undo' the calculation *is* 'unknown'.

Askew (2003) describes the Realistic Mathematics Education process of vertical mathematizing as 'recasting the model' or working with symbols. Once the 'context' has been stripped away there is the possibility of working with the arithmetic model freed from the influences of the problem-statement. Because of the property of commutativity, 25 could be considered as the addend and the model could be 'recast' as

$$? + 25 = 41$$

and subsequently use of subtraction as the inverse of addition may be more noticeable. Hence the arithmetic model can be further recast as

$$41-25= ?$$

Askew identifies that some of the possible stages of vertical mathematizing are implicit but does highlight that none of the children he observed in a visit to a Year 3 class seemed to 'recast' their initial models through vertical mathematizing. Alan's behaviour is consistent with this observation. Judging from his description of his thinking it does seem clear that he identified the problem-statement as an augmentation situation. The approach he adopts is one of using trial and adjustment to work out the amount that needs to be added on. He appears flexible and confident in the way that he makes an estimate and successively adjusts this. He talks as though he had a very clear idea of how to work on the problem.

In adopting a trial and adjustment approach he is working directly on what seems to be the initial arithmetic model that he formed and does not manipulate this in any way. There is no obvious evidence from the interview to identify why he took this approach rather than one that involved vertical mathematizing (manipulation of the model). However a conjecture arising from this discussion is that the 'context' described by the problem-statement implied an augmentation structure of addition and that recognition of this structure may have affected the strategy chosen to work on the problem.

Alan worked on solving two other similar augmentation problems during the study. Each of these identified the addend as the unknown that needed to be found. On each occasion Alan produced recording that suggested that he was thinking about how much had to be added on to the augend to make the total. There was no indication of any attempt to 'recast' his initial model of the problems (Figure 4.1.7).

Figure 4.1.7 **Solving Tasks** **Alan**
Problem I MonInc2 Ann had £28. Her parents gave her some money for her birthday. Now she has £41. How much money was she given for her birthday? <u>Response</u> 28 + (13) = 41 **Problem C BusInc2** On a bus there were 35 people. At the bus stop some passengers got on. There were now 49 passengers on the bus. How many people got on the bus at the bus stop? <u>Response</u> 35 + ? = 49 35 + 14 = 49

It seems that Alan produced an accurate recording of these problem-statements using appropriate symbols but did not go on to work with the number-statements as tools that might have aided him in coming to an efficient solution. Having identified an appropriate symbolic representation he did not manipulate it as part of his strategy to solve the problem.

This pattern of response however was not evident with other problem structures. Problem W (Figure 4.1.8) is also an augmentation situation but differs from the problems above in that it identifies the augend (rather than the addend) as the unknown to be found.

Figure 4.1.8 Solving Task **Alan**
Problem W Sue had £27 after she had been given £12 by mum. How much money did she have before mum gave her the money?

If the pattern of Alan's behaviour described above were to be repeated we might expect him to produce an initial model of Problem W that looks like this

? + 12 =27

and then for him to provide an estimate of the unknown and work on this by trial and adjustment. However the recording that he produces is

27 – 12 = ?

This particular form of recording might be expected after manipulation of the symbols of the initial arithmetic model. He may have implicitly devised an initial model rather as Askew suggests and not recorded it or he may have in some way moved directly to the calculation that he thought he needed to do. He explains his reasoning in the following way:

> "... because it is telling you that Sue has £27 after she has been given £12 so you have to take it away to find the answer"

There is perhaps a hint here of comprehension of the augmentation structure of the problem-statement. So mentally he may have constructed and manipulated an initial model. However as Alan's recorded comments about his thinking come after he has solved the problem not during this thinking, the comments might reflect *post hoc* justification rather than 'in the moment' thinking. It is perfectly possible that he saw the problem as requiring subtraction of 12 from 27 in an instant and worked directly with this. In my experience many learners, including many adults, in some way 'see' problems like this one as a subtraction situation and only with some assistance come to see the distinction between this type of problem (an augmentation situation that requires a form of subtraction in order to solve) and a more direct takeaway situation (starting amount (minuend) and amount subtracted (subtrahend) given, the result (difference) as the unknown). In such cases there seems to be no explicit or even implicit manipulation of their initial mathematical representation of the problem-statement. The learner seems able to 'jump' straight to the calculation needed to solve the problem without the need to consider the model implied by the problem. Behaviour such as this might indicate Alan's level of awareness as being consistent with Mason's (1998) *awareness-in-action,* a phenomenon which will be further discussed later in this section.

However we account for Alan's working on Problem W it is curious that he seems to respond differently to augmentation problems that differ in little other than whether it is the augend or the addend that is unknown. In a sense he is displaying a fine distinction between problem types. It is not a distinction that is evident in the objectives and guidance that the PNS Framework for Mathematics provides on working with word problems nor does it seem to be at all obvious to teachers and trainees with whom I have worked. In my experience addition and subtraction word problems seem to be viewed by many colleagues as having only two possible categories, those that can be solved by addition and those that can be solved by subtraction.

In considering Alan's mathematical development should we be satisfied with the behaviour he is currently exhibiting, which is quite impressive as it stands, or should we consider how we could influence his awareness in order to help him to make connections that will enable him to develop more efficient strategies? Specifically we may want him to become aware of the possibility of manipulating the initial models that he develops rather than working on them directly. Thus it would be useful to further develop his ability to think 'proceptually' in the sense of Gray and Tall (1994) so he becomes more flexible in his response to problem statements. In particular it would be useful for Alan to 'switch' from viewing the augmentation model of addition implied by the problem statement to using an 'inverse of addition' model to carry out an appropriate operation. In doing so we may need to provoke his awareness in such a way as to try to initiate a shift from *awareness-in-action* to *awareness-in-discipline* (Mason, 1998).

The nature of tasks that might influence his awareness appropriately and the implications this has for teacher subject knowledge will be discussed subsequently.

4.1.4 Barry

Barry appeared to use an appropriate operation when attempting to solve problems presented to him. He also tended to use similar approaches to each operation but did not always carry them out accurately. A particularly striking feature of his work is the inconsistency of his performance. This was first identified in considering his responses to the February and July probes (Appendix 7: February and July Probes for Barry). Several probe items that were correctly answered in February were answered incorrectly in July. This general phenomenon reflects unstable behaviour and should perhaps also include examples such as those probe items that Barry answered incorrectly in February but correctly in July. The latter is of course a desirable change in behaviour but does represent an inconsistency and so could be considered as unstable behaviour. Subsequent analysis of the data relating to the sessions that Barry took part in revealed a significant proportion of unstable behaviour and an attempt was made to classify each particular example of such behaviour.

The most obvious form of unstable behaviour seemed to be where different answers were given to the *same items presented on different occasions*. This could be illustrated by Barry's answers to number-statements. He produced different answers for four items out of the seven on the probe administered in February and July (Figure 4.1.9).

Figure 4.1.9 February and July Probes (Appendix 7) Barry			
Item	February	July	
A	41	31	Shaded cells indicate incorrect answers
C	34	32	
E	17	26	
F	15	25	

In each case the item was correctly answered in February and an incorrect answer was given in July. A possible rationale for each incorrect answer can be constructed although as the probe was administered in a 'test' format no interviewing took place and these accounts are speculation. Less obvious but perhaps more interesting would be accounting-for why items answered correctly on one occasion were answered wrongly on another.

There were also three probe items relating to problem-statements (out of a total of seven) where there was a different answer given on the two probe dates (Figure 4.1.10).

Figure 4.1.10 February and July Probes (Appendix 7) Barry			
Item	February	July	
B	42	32	Shaded cells indicate incorrect answers
D	39	32	
F	25	15	

In Problem B the change was from correct to incorrect whereas on Problems D and F the change was towards that of a correct answer. It is interesting to note that although each number-statement was designed to have a corresponding problem-statement there was at least one incorrect answer for each pairing across the two presentations (Appendix 7: February and July Probes for Barry). This comparison is the source of the next class of unstable behaviour.

Looking at Barry's answers to the corresponding number and problem-statements suggested another form of unstable behaviour. Word problems are considered by some authors (Gerofsky,1996; Lave,1992) to be little more than calculations embedded in words. If this view is taken, then an observed form of unstable behaviour involved *different presentations of the 'same' mathematics closely related in time*. A particular example of this is the number and problem-statement pairing D presented in February (Figure 4.1.11).

Figure 4.1.11 February Probe Pairing D Barry			
□ – 5 = 27	32	**D Change 6** There are 27 children left in the class at home time. During the day 5 children had gone home because they were ill. How many children were present at the beginning of the day?	39 *27+5=12* *27+12=39*

The number-statement is answered correctly but the problem-statement has an incorrect answer despite an appropriate calculation being chosen. On other occasions such as the pairing A presented in July the number-statement is incorrect while the problem-statement is correct (Figure 4.1.12). In total there are three examples of the former and four of the latter.

Figure 4.1.12 July Probe Pairing A Barry			
16 + 25= □	31	**A Change 1** Shahina has £16. She is given £25 more by her parents. How much does she have now?	41 *16+25=41*

Despite requiring the 'same mathematics' to solve each aspect of the pairing, solving a problem might be considered to be very different to finding the unknown in a number sentence. However it is of some interest that incorrect responses are balanced between the two formats and that on two occasions an incorrect answer to a problem results from a computational error (Appendix 7: Probes for Barry). It could be that a different approach to a calculation may be taken when that calculation has resulted from reading a problem. If this is the case then the stance taken by Gerofsky (1996) and Lave (1992) that word-problems are little more than calculations embedded in words could be questioned as Barry appears to respond differently to each situation. However the data from Barry's probes only involves his recording and it is not clear from this if a different approach has been taken in any presentation of any pairing.

There seems to be some inconsistency in Barry's response to *similar number-statements involving different numbers* of the same order. For example in July the number-statement A (16 + 25= ?) is answered incorrectly as 31 whereas the number-statement B (23 + 19 = ?) is answered correctly. These are very similar situations and each would perhaps be considered to be well within the capability of a Year 4 child such as Barry who rated above average by his teacher. There is also inconsistency between the number-statements C and D in July (Appendix 7).

The *order of mention of events* in a word problem has been well documented (Teubal and Nesher, 1991) as affecting performance. Problems where the order is not consistent with the chronology of events tend to generate more incorrect answers. In producing an incorrect answer to Problem D in February and a correct answer to Problem C Barry's performance is consistent with this general phenomenon. There are other examples of this evident in Barry's data such as with problems A and B in July.

Problems E and F are combine problems and so don't imply a chronology. However they are very similar problems except that they differ in the *order of mention of variables*. Problem E which states the 'total' first and then the known subset is answered correctly by Barry whereas problem F which has the reverse order is answered incorrectly. So *order of mention of variables* seems to affect Barry's performance even when chronology is not a feature of the problem-statement.

So how can Barry's unstable behaviour be accounted-for? It could be that Barry's awareness is of mathematics being unprincipled. Such awareness would not appreciate the structures, patterns and consistencies of the subject; hence there would be no rationale for expecting stability in response to tasks. A learner with such awareness might be unduly influenced by the first thought that comes to mind. Another cause of the instability could be a lack of engagement or motivation on the part of the learner. The randomness of response that this accounting-for suggests is in conflict with the view that many errors that learners make in mathematics can be explained by logical, though flawed thinking (Ginsburg ,1977); van Lehn, 1990). This provisional accounting-for will be developed further in Chapter 5 where changes in the children's behaviour over time will be the focus of analysis.

4.1.5 Further discussion

In addition to unstable behaviour several other themes were apparent when considering the data from the problem-solving tasks. Although each theme seems to relate mostly to the behaviour of a particular child it is sometimes possible to find examples of the phenomena in the data relating to other children. Identified themes include:

- A tendency to respond to problems in a way that suggests a default 'trigger' response to addition;
- A tendency to respond in a way that suggests attention is focused on a particular word or phrase in the problem-statement;
- A tendency to use different calculating strategies according to perceived distinctions in the structure of similar problems;

In researching an area of mathematics education such as word problems it is reasonable to be seek out material that directly relates to that area. It is also reasonable and profitable to consult more general material from the mathematics education literature, hence the discussion of material of this kind in the literature review (Chapter 2). In the course of any enquiry it is likely that analysis may draw attention towards material that, prior to the study, may have not seemed directly relevant. This has been the case in this study, hence in the preceding sections and in the discussion that follows further material that resonates with themes arising from the data analysis is introduced. Some of this material clearly relates directly to mathematics education but it has been fruitful to explore other material that on the surface may not appear to be relevant to mathematics education but with deeper examination usefully informs the discussion.

Information Processing and Dual Process Theory

Minsky (1974) develops the idea of representing knowledge as a collection of *frames*. He suggests that when an individual encounters a new situation a structure called a *frame* is selected from memory. Frames tend to represent stereotyped situations and are developed as a result of previous experience. However they can be adapted by an individual to fit the current situation. Contained in a *frame's* structure is information about how to use the *frame*, what one can expect to happen next and what to do if these expectations are not confirmed. Davis (1984) has drawn on frame theory to account-for some of the phenomena that are evident in learners' responses to mathematical tasks. In particular he suggests that although the collection of ideas (frames) in any mind are unique to that individual, there are a number of commonalities that might be shared in nearly identical form in the minds of different learners. He identifies several specific frames that are useful in accounting-for common inappropriate responses.

Karen's tendency to add the numbers presented to her in a problem-statement may be an example of one of Davis' frames in action. He explains that the *undifferentiated binary-operation frame* is a response appears to be shared by many learners arising almost inevitably from their previous experience of addition. He argues that in the early years of school children experience extensive drill in addition facts and that as a result of this 'a very permanent and stable representation of addition ... ... is created' (Davis, 1984, p 111) in the mind of the learner. Furthermore as addition is the only operation that a child is likely to encounter at this stage there is no stimulus to prompt him to discriminate between operations. So effectively at this stage a child is perhaps only responding to the numbers presented in an addition fact and has no need to take account of the addition symbol. Davis also argues that *frames* in an individual's collection are only added to, none are deleted so this particular response will always be there as a part of the learner and will compete with subsequently developed frames for retrieval.

There are perhaps two ways in which the *undifferentiated binary-operation frame* can be used to account-for Karen's behaviour. Davis suggests that the frame 'accepts as inputs the non-negative integers *a* and *b* and, ignoring any operation signs, returns *a+b'* (Davis, 1984, p 111) In the same way a problem-statement may be seen as an input in which any implied meaning of the words is ignored and the learner performs an addition operation on the numbers presented. This description suggests a quick, default trigger-like response from the learner in which little or no consideration of the problem-statement as a whole is made. An alternative explanation is that the whole problem-statement is considered in detail but the learner is unable, as a result of this, to plan an appropriate way forward. However Davis suggests that rather than grinding to a halt if a goal cannot obviously be attained a lesser goal is settled for by the learner. This lesser goal is seen as the better of the possibilities available. This he describes as the *highest-level programs must run* principle of human information processing. In Karen's case rather than providing no answer she uses the best option available to her. As previously suggested the *undifferentiated binary-operation frame* ensures that this option is the use of addition. The latter account is consistent with the perspective that van Lehn (1990) developed when investigating children's errors. He suggests that human beings find ways of working around problems (impasses) that arise rather than ceasing to function. However the motive behind the activity drawn on to overcome the impasse is often to follow a procedure as closely as possible rather than achieving a specified final state (the correct answer). Thus Karen's motive might have been to carry out some mathematical operation to produce any answer rather than to select the appropriate operation to produce the correct answer.

These alternative explanations have parallels with Leron and Hazzan's (2009) discussion of dual-process theory. This theory suggests that human cognition and behaviour operates in two different ways. The first of these, System 1 (S1) tends to be intuitive in nature and is "characterised as being fast, automatic, effortless, unconscious and inflexible (hard to overcome)" whereas System 2 processes are "slow, conscious, effortful, computationally expensive (drawing heavily on working memory resources), and relatively flexible". Leron and Hazzan suggest that S1 can "hijack" a learner's attention and prevent S2 processes from monitoring and intervening in appropriate and beneficial ways. They go on to suggest that an important implication for education is a "need to train people *to be aware of the way that S1 and S2 operate, and to include this in their problem solving toolkit*".

If the quick, default trigger-like view of Karen's behaviour is taken then she would seem to be drawing on S1 processes. However the alternative account suggests the involvement of S2 processes in her consideration of problem-statements. Whether we consider Karen to be working in S1 or in S2 it still seems that Davis' frames and Leron's Systems may inform each other in accounting-for her behaviour. For example if we take an S1 view of Karen's response then Davis' *undifferentiated Binary-operation frame* may explain why she responds with addition rather than any other possibility while the *highest-level programs must run* principle may be a feature of the S2 reasoning. It might be that dual-process theory describes a general account-of reasoning in this case while Davis' frames and information processing principles describe a little of the specific detail of the 'mechanisms' involved.

So far the analysis has attempted to account-for observed phenomena by focusing on the psychology of the learner. However word problems could be considered as objects with their own psychology. An analysis from this viewpoint may inform the discussion. The next section introduces this idea.

Word Problems as Objects

Norman (1988) in his book *The Psychology of Everyday Things* suggests the existence of a psychology of objects. Everyday objects are seen as having perceived and actual properties that determine to some extent how they may be used. The possible actions suggested by the properties of an object are referred to as *affordances.* In the following example he explains the affordances of some commonly used materials:

> A chair affords ('is for') support and, therefore affords sitting. A chair can also be carried. Glass is for seeing through and for breaking. Wood is normally used for solidity, opacity, support or carving. Flat, porous, smooth surfaces are for writing on. So wood is also for writing on. Hence the problem for British Rail: when shelters had glass, vandals smashed it; when they had plywood, vandals wrote on and carved it. The planners were trapped by the affordances of their materials.
>
> (Norman, 1988, page 9)

Thus affordances of objects provide clues for their use and sometimes that use is undesirable. According to Norman there are other aspects of the psychology of objects that influence their use. He explains how the holes in a pair of scissors are affordances that suggest fingers should be placed through them but that the size of the holes is a *constraint* which limits the number of fingers that may be used. Thus an affordance is a suggested action or use while a constraint is some sort of limit on that use. The ideas of affordances and constraints were originally developed by Gibson (1977, 1979) and introduced to Mathematics Education by Greeno (1994) who explains that:

> "In any interaction involving an agent with some other system, conditions that enable that interaction include some properties of the agent along with some properties of the other system... ... the term affordance refers to whatever it is about the environment that contributes to the kind of interaction that occurs"
>
> (Greeno, 1994, page 338)

Greeno explains that Gibson saw affordances as invariant, they do not change as the needs of the user change, they may not even be perceived by the user. So glass always affords "seeing" and "breaking" and seems likely to have other yet to be perceived (by this observer) affordances. Objects though do not exist in a vacuum and considerations of interactions with them have to take account of the properties of the agent. Brown, Stillman and Herbert (2004) explain this as the object having attributes and the agent or actor having particular *abilities*. These abilities are sometimes referred to as *attunements* and might be thought of as the prior knowledge, experience and sensitivities that a learner brings to an interaction (NCETM Online). Thus any interaction between an object and an agent is influenced by the affordances offered by the object and the attunements that the agent brings to the situation. The attunements brought to an interaction with a window are likely to be very different if the agent is a cleaner rather than a vandal and may result in very different responses. Before further developing this psychology of interaction it may be worth starting to identify its relevance to my interest in learners working with word problems.

My starting point is to take the stance that a word problem is an object and that as an object it has its own psychology in a similar way to the everyday objects that Norman describes. Hence word problems have properties that *afford* particular actions by learners. At a general level a word problem affords ("is for") attempts to solve it or at least attempts to provide an answer. Of course other tasks with word problems can be designed, for example the sorting, matching and constructing tasks used in this study, though in classroom practice solving tasks seem to dominate. This is a point that will be further developed shortly.

As well as affording attempts to solve them, the nature of word problems may also afford the use of particular types of activity in these attempts. In particular they afford the opportunity to mentally enter a situation, detect relationships and engage in actions informed by these relationships. Word problems have been described as having particular features including format, context, semantic structure, and mathematical structure. Some of these features such as the mathematical structure are not easily perceived by learners while others like the words and numerals in the format are more obvious. The nature of these features affords particular actions by learners encountering word problems though it seems that in interactions the affordances offered by some features seem to dominate those offered by others. For example in this study much activity by learners seemed to be centered on numerals and keywords. The focus on numbers may be partly accounted-for by their obviously perceived nature. In most cases the numbers in word problems are usually written as numerals and so stand out from the words in the problem-statement thus affording activity focused on them. An interesting observation is the lack of a response from Karen in the initial probe to the one word problem that presented numbers expressed as words rather than numerals. This particular problem although only slightly different in presentation seems to offer different affordances to those offered by the other problems. The presence of numbers represented in numeral form may have afforded Karen some activity involving numbers whereas numbers presented as words did not. Depriving Karen of the numerals may be akin to presenting someone with a pair of scissors that have no holes for fingers. In each case it is not clear what you do with the object!

Other children did provide answers to 'Karen's problem' and hence responded differently to the 'same situation' so it seems that there is more to consider here than the affordances offered by the properties of the word problem. Clearly each learner brings their own attunements to the interaction. Each will have had different prior experiences and will have developed different sensitivities. As a result they may respond differently to the affordances offered by a particular problem. Another way of accounting-for Karen's behaviour is to consider the experience that she brings to the task. Her particular attunements may have played a role in her responses. She may have been attuned to identifying and responding to numbers in numeral form but not in word form. A particular action can only result if an appropriate relationship between affordance and attunement exists. Such a relationship appears not to have developed for Karen.

There are more influences on any interaction between object and agent than the factors so far considered. Objects and interactions sit in time and space and in doing so become part of a system rather than isolated elements. Objects in particular systems may offer affordances that don't exist either in isolation or in other systems. It may even be beneficial to think of the system as offering affordances rather than the individual objects. Different systems that may contain the same object are likely to provide different affordances. An agent encountering a system containing a window and a hammer in close proximity is offered some different affordances to an agent encountering a window and a cleaning cloth. In a similar way a word problem presented in isolation affords different possibilities of action to the same one presented immediately following an exercise of calculations or one presented in the middle of a test paper. In each of these cases the same object (word problem) could be considered to be part of a different system and so provide different possibilities. The use of tasks involving the sorting of problem-statements is another system within which word problems might be used. The intention with these tasks is that learners are afforded the opportunity of becoming attuned to the importance of words and relationships expressed in the problems. Children's responses to these tasks are presented and discussed in the next section (Section 4.2).

In discussing affordances and attunements I find myself encountering a particular dilemma. I can only specifically react to or discuss any affordance that I can perceive so when I state that the systems above provide different affordances I can only give examples that I can perceive. These perceptions result from the attunements I bring to the situation so the particular affordances that I discuss are only those that I am attuned to notice. Does a word problem with numerals really offer different affordances to one where the numbers are expressed in words? I'm not sure, but it does seem a strong possibility that Karen's failure to produce an answer, on the two occasions when word problem statements expressed numbers in written words, could have been a result of her particular attunements. These attunements might be such as to not allow her to access the affordances offered by problems where numbers are represented in the form of written words. It is becoming clear that as well as considering the psychology of the object and the psychology of the agent it is necessary to consider the psychology of the system as a whole including the role played by people and cultures.

Norman (2007) explains that in his earlier work his reference to affordances was to perceived affordances and that in situations that involve people in some form there are affordances and constraints but there is more. He is alluding to trails and behaviours, explaining that we often know what to do by watching the behaviour of others or by the evidence or trails that they leave behind. Norman views people as detectives who are searching for clues about how to act. He gives examples about how we might decide whether our train has left by the number of people on the platform or which way the wind is blowing by the motion of a flag. Significant signs like these arise in the world around us and offer guidance. These cues he calls social signifiers:

> "A "signifier" is the some sort of indicator, some signal in the physical or social world that can be interpreted meaningfully. Signifiers signify critical information, even if the signifier itself is an accidental by-product of the world. Social signifiers are those

that are relevant to social usages. Some social indicators simply are the unintended but informative result of the behaviour of others."

(Norman, 2007, Online)

Norman calls any perceivable cue a signifier explaining that they may be deliberate or incidental. Social signifiers are created or interpreted by people, signifying some social behaviour. They are often of use, but cannot always be relied upon. Signifiers seem to take account of the roles of affordances *and* attunements in an interaction. For a feature to be a signifier it has to be an affordance and the agent needs to be attuned to it.

Word problems are designed by authors and teachers who in the construction of a particular problem are likely to include clues to the learner about how to proceed. Sometimes this is done very deliberately as in the case of problems in SAT tests where keywords and numbers are written in bold type. Such clues could be considered to be signifiers as they give guidance about the way forward. Sometimes signifiers are not so explicitly presented and may not be intentional. For example, Sowder (1988) explains that some children respond to word problems by looking at the numbers and choosing an operation that seems appropriate. In the problem below it may seem unlikely to a primary school child that they would be expected to use division or subtraction with these numbers and addition may seem too easy, hence the presentation of the numbers 8 and 7 might signify multiplication

Paul has 8 shirts and 7 pairs of shorts.

How many different outfits can he make up from these?

It is unlikely that the author would have intended the numbers to act as a signifier in this way. In this particular problem multiplication is the appropriate response but in another problem containing the same numbers it might not be, perhaps identifying the mixed reliability of signifiers. So word problems provide clues (signifiers) about how to act. The clues may be deliberate on the part of the author or may be incidental. They may be useful in helping a learner to produce an answer but unhelpful in developing the thinking of a learner. The previously cited example of a word problem being preceded by a calculation exercise based on a single operation signifies that the same operation should be used to solve the problem. A page in a textbook I once used was entitled *addition problems*, signifying that addition needed to be used to solve the problems. In each of these cases the signifier aided learners in providing a correct answer but may have done little to help them to develop more useful general strategies that might be used when these particular signifiers are not present. The milieu of many classrooms provide signifiers that children may use. In Butlin (2003) I explained how use of the keywords *more* and *less* by a teacher in dialogue and on displays seemed to signify the use of inappropriate operations by children solving word problems. Approaches that involve making learning objectives and success criteria explicit seem designed to provide learners with signifiers for actions they may take. If not carefully chosen and expressed the learning objective may provide signification that is unhelpful to a learner's long term development. In this vein I particularly remember a teacher who used the expressed learning objective "To solve multiplication problems".

This signified the action that the children needed to take to find the answers so there was little need to engage with the problems themselves except in the most superficial of ways. These examples tend to present an image of signifiers as hindering development of appropriate thinking but this is not true of all signifiers; perhaps the role of the teacher is to aid learners to develop strategies that attune them to become aware of more appropriate signifiers in a situation. The aim of the sorting, matching and constructing tasks in the following sections is to do just this.

Norman (2007) wrote of the clues to be gained by observing the behaviours of others and using the trails they leave behind. Teachers often model and demonstrate actions to be followed in particular situations and to some extent they leave trails by their feedback and displays. Teacher behaviour is often thought out in advance, although there are instances where teachers have to act in the moment. Plans are made (or found) and these are followed. The behaviour that arises out of these plans is the classroom version of that which Norman describes. It is this behaviour and the trail that it leaves that may guide learners about how to act. Thus it is important that teacher behaviour is appropriate and useful. In particular teachers need to ensure that they are aware of their behaviour and of the effects of any inappropriate signifiers it might highlight. The implications for the development of teacher subject and pedagogical knowledge arising from this will be discussed subsequently.

The argument so far considers word problems as objects that offer affordances. Learners come to particular word problem tasks with personal attunements. The amalgam of affordances and attunements provides signifiers that guide a learner in moving forward in the task. For any individual there may be a range of possible signifiers, so what informs the learner to 'choose' or be drawn to one signifier rather than another? Consideration of the idea of metonymy may begin to address this question.

Metonymy

Lakoff and Johnson (1980) describe metonymy as the use of "one entity to refer to another that is related to it' (p35). They explain that this relationship may be of a part-whole nature in an example like "I've got a new set of wheels" where one part, in this case the 'set of wheels' is used to stand for the whole car. Another type of metonymy is part-part. Here one part of a conceptual structure is used to refer to a different part of the same structure as in "The strings played superbly". In this example 'the strings' is used to refer to 'the players' with both 'strings' and 'players' being parts of a larger structure, the orchestra. Metonymy as well as being a regular part of our day to day communication can also be found in the practices of mathematics. For example Pressmeg (1997) has identified situations where "one element of a class may be taken for the whole class'. She explains that this has been referred to as a paradigmatic model and includes examples such as sketching one triangle to represent a whole set of triangles.

Metonymy may have a significant effect on a learner's activity when working on a mathematical task. At a general level Zandieh and Knapp (2006) point out that the use of one part of a structure to stand for the whole draws attention away from other parts. There may be little consequence in attention being drawn to the 'set of wheels' and away from the doors or any other feature of a car but there might be some consequence if metonymy draws a learner's attention away from some important feature of a mathematical structure. Zandieh and Knapp (2006) further explain that usage of a particular feature of a whole structure or a particular example of a whole set may be the choice of the speaker and that this shows his or her focus. However they also explain that this 'choice' may not be neutral. There may be cultural influences that almost predefine the speaker's choice. The power of such influence should not be under-estimated as an example from my own experience serves to illustrate. I have asked thousands of trainee teachers to draw a hexagon on a small hand-held white board and then hold it up for me to see. In every single case the trainee has attempted to draw a regular hexagon. I have found this response quite striking and have accounted-for it by the very high frequency of use of regular hexagons in text books and classroom displays. These trainees may have had little real choice in their response as their experience of hexagons is likely to have been dominated by exposure to regular examples of the shape. In its use in the mathematics classroom (and the wider world) the regular hexagon has become a metonymic representation for the whole class of hexagons. This usage clearly has a strong influence on learners but there are possibly other factors that have contributed to such a strong learner response. Lakoff (1987) provides an outline of Rosch's work on prototype effects. Rosch identified that not all members of a set have the same status. Some members seem to have "a special cognitive status-that of being a 'best example'". That is particular items, prototypes, may be more likely to be chosen as an example of a set than others. This asymmetry manifested itself in several ways in Rosch's data including by direct rating, reaction time, production of examples and generalisation. It would seem that regular hexagons may be prototypes in this sense thus exerting a further influence on the responses of my trainees. It may be that particular features of a regular hexagon cause the shape to assume the status of prototype and then usage by teachers and authors provides a level of exposure that further strengthens this prototype in a way that is similar to that described by Davis (1984) in presenting the *undifferentiated binary-operation frame*. So although, in principle, my students had a choice of which hexagon to draw, in practice the influences on them were so great that they had little 'choice'.

So what has metonymy got to do with learners working on word problems?

Any particular word problem-statement is in itself a conceptual whole made up of a number of 'parts'. Previously we considered Verschaffel, Greer and de Corte's (2000) analysis indicating that mathematical structure, semantic structure, context and format are all component parts of a word problem. There are other features such as the words and numbers visible on the paper that are literally surface features. Labels for people and objects may also be considered surface detail, while components such as the described context and the implied mathematical structure are deeper less obvious features. It has been argued earlier that Karen may be focusing on the numbers in the problem-statement and responding to these by using addition. It may be that this response is a type of 'metonymic trigger'. This is not metonymy in the sense of a speaker communicating but it might be metonymic reasoning. Rather than considering the whole conceptual structure of the problem-statement Karen considers just one part of it (the numbers) and responds to this. This sort of reasoning involves part-whole metonymy, a part of the whole problem-statement is used to stand in for or represent the whole problem. It might be that that this focus on numbers is a clear choice on Karen's part but it could be that she has little real 'choice' in what to attend to just as the trainees drawing regular hexagons had little 'choice' due to the strong influences on them. Numbers are clearly different from words and so stand out from the words in the problem-statement particularly in the context of a mathematics lesson so making them readily discernible. As a component part of a problem-statement numbers perhaps acquire prototype qualities for some learners like Karen. In the way that 'wheels' provide a more obvious metonymy for a car than doors so numbers provide a more obvious metonymy than the words in a problem. When a metonymic attraction of this sort is established then Karen's attention may be drawn away from other features of the problem-statements that might be more helpful to her. Furthermore Karen might have developed a metonymic connection between numerals and addition, of all the actions that can be carried out with numerals addition might be for her the prototype or most obvious to be drawn to.

In this analysis there seem to be connections between metonymy and the processes of System 1 reasoning. Metonymic triggering may be a mechanism which System 1 processing draws on while Davis' *undifferentiated binary-operation frame* may explain the particular activity that results from the triggering. The possible metonymies offered by a word problem are part of the set of affordances that it provides. Metonymies might be idiosyncratic and contain an emotional aspect resulting from a learner's previous experience. Those metonymies that Karen is attuned to become signifiers that guide her action but in being drawn to one particular signifier rather than another she is influenced by the effects of prototypes and frames which could be considered to limit her real 'choice'.

Building on the use of metonymy in analysing Karen's behaviour it may be fruitful to consider Lakoff's (1987) general metonymic model of reasoning which as will be discussed, seems to provide parallels with children's observed responses to word problems. He describes this general model as having the following characteristics.

> There is a 'target' concept A to be understood for some purpose in some context. There is a conceptual structure containing both A and another concept B.
> B is either a part of A or closely associated with it in that conceptual structure. Typically, a choice of B will uniquely determine A, within that conceptual structure.
> Compared to A, B is either easier to understand, easier to remember, easier to recognize, or more immediately useful for the given purpose in the given context
> A metonymic model is a model of how A and B are related in a conceptual structure; the relationship is specified by a function from B to A
> When such a conventional metonymic model exists as part of a conceptual system B may be used to stand, metonymically for A.
>
> (Lakoff ,1987, pages 84-85)

It is clear to see how such metonymic reasoning may result in incorrect answers if applied to solving word problems. If Karen's behaviour is interpreted using this model the target concept may be the mathematical structure of the problem-statement. The numbers in the problem-statement are associated with this structure and are easier to recognize, understand and remember than the mathematical structure and so come to be used in place of this. Numbers or specifically numerals become signifiers that guide Karen's subsequent action. Karen's response to the metonymic trigger of numbers seems to be to use the operation of addition as suggested by the *undifferentiated binary-operation frame*. In this case it seems entirely inappropriate and unhelpful that a feature such as 'numbers' should stand in place of a concept such as 'mathematical structure'. However 'numbers' are not the only feature of word problems that seem to be used in a metonymic way. Other children in the study seemed to respond to 'keywords' or short phrases in the problem-statements so using them as signifiers to guide their actions. For example in the second session Haleema suggested that the words 'got on' influenced her to use addition to solve a problem. On this occasion a correct answer resulted. However in the third session she used addition again saying that the phrase '*people got on the bus*' influenced her choice of operation. On this occasion addition was inappropriate, thus illustrating the unreliability of short phrases of problem statements as appropriate signifiers for action.

This too might be considered to be metonymic reasoning in that a part of the problem-statement "got on" is being used to stand in for the whole problem and a response is generated taking into account only this phrase. Such metonymic reasoning might occur naturally as children work on word problems but some of the approaches that teachers take to supporting children with word problems may encourage such reasoning. In particular the advice to underline keywords and numbers may encourage attention to be drawn away from more helpful components and produce metonymic type consideration of these unrepresentative features.

This discussion seems to suggest that the use of metonymic reasoning alone as an approach to solving problems is faulty and at best will produce correct answers for the wrong reason. If metonymic reasoning has a tendency to be inappropriate for problem-solving then what sort of reasoning might be appropriate and how do we develop it? This will be discussed in the next section. However it seems that to develop such reasoning any negative effects of metonymy need to be overcome. Attention needs to be drawn away from less helpful signifiers to other aspects of a conceptual structure that may provide more appropriate guidance. This may involve a learner delving deeper into a problem-statement and coming to understand the problem as a whole.

Awareness and attention

As has been mentioned in Chapter 2 Mason (2004) feels that what is important in a task is 'what learners are attending to and how'. The behaviour outlined in the three themes identified at the beginning of this discussion might suggest different foci of attention. Karen and Haleema in appearing to respond to numbers and individual words or phrases are attending to surface details whereas Alan provides some evidence of responding to the mathematical structure of some problems, a deeper and less obvious feature. In this analysis I have no intention of implying that one child is paying more attention than another or is in any way working harder; or that learners are paying more or less attention or working more or less hard on different occasions. It seemed that on every occasion children were well motivated and trying very hard. I am thinking more about which aspects of the problem naturally draw the learner's attention at a particular time.

These claims about each learner's focus of attention are being made after a learner has worked on a task. Sometimes the evidence is their recorded work, sometimes it is post task discussion. An aspect of their attention that this evidence does not reveal is if the focus of attention has changed prior to a final response being given to a problem. Another aspect that is not clear is the form or level of the learner's attention. However, we may speculate about this. Alan, for example, does not seem to be giving his explicit attention to differences in the mathematical structure of problems despite his behaviour seeming to respond to such differences. This may be an example of *awareness-in-action*. On different occasions there seems to be some evidence of Haleema attending to different foci but it is not clear if these shifts are occurring frequently within sessions. In each case an appropriate way to further develop the learning of these children may involve some task designed to provoke them to become more explicitly aware of their attention and begin to build *awareness-in-discipline*.

Looking at this another way, perhaps adopting a teacher's viewpoint, we may want to influence a learner to attend to aspects of the structure of a problem rather than only to key phrases or just numerals. There is a suggestion in the data that Karen's attention had begun to shift from numerals to words and that it would be beneficial if Haleema's attention shifted to considering structure rather than phrases. It may be useful to Alan to notice a connection between the addition number-statements that he writes and subtraction. Basically we perhaps want them to notice and attend to some aspect that they previously seemed not to notice. A shift of this kind could be considered to be the sort of transformation that Mason and Johnston-Wilder (2006) refer to when considering the nature of learning. However such changes could be very subtle and may go unnoticed by teachers who are required to attend to a wide range of events in the course of their daily work. Indeed it is only some time after the collection of data and after perusing it many times and in different ways that some of the identified phenomena became apparent to me.

A shift in attention, although in itself evidence of learning (Mason and Johnston-Wilder, 2006), may not result in a child performing in an improved way when assessed against the success criteria so commonly used in English primary classrooms. A change of focus from key words to considering the whole problem-statement may still result in incorrect answers. On the other hand an improvement in performance against success criteria may not be the result of any desirable shift in awareness. This would suggest that identifying transformations in perception can be problematic in the normal milieu of the primary classroom and so raises questions about the common approach to teaching that seems to require short term attempts at diagnosis and remediation.

Tasks involving the solving of word problems tended not to provide me with ways of readily identifying the aspects of a situation that a learner was attending to or the nature of the activity that they were engaging in. Answers and recording of calculations were only helpful with much speculation. Even talking to children about their activity when working on problems seemed to reveal little. In fact considering the nature of the discussion that occurred with children after they had worked on a problem revealed another theme which all of the learners displayed. This was the tendency to describe the approach to the calculation they used with little or no comment on any other activity that may have taken place. This may suggest a lack of awareness on their part of this activity or of its importance. As all of the sessions with me were likely to be seen as 'maths' then it is perhaps not surprising that 'manipulation of numbers' dominated discussion. The development of tasks which might facilitate a shift in focus from discussion of calculation to other aspects of problem-solving activity is likely to be of value to a researcher but may also have a role to play in developing a learner's mathematical ability. If a particular task helps a learner to talk about the wider activity they engage in then it seems likely this would increase their awareness of such activity and so exert an influence on the learner's propensity to develop and work in such ways.

So what sort of task would be useful in trying to encourage a shift in a learner's level of awareness and a change in the form and focus of their attention? Whilst not dismissing the use of tasks that involve the solving of problems, with some feedback about success, it may be that some other form of task might add positively to a learner's experience if used in addition to the solving of word problems. 'Guidance' on the possible nature of such tasks is available in the literature but in my experience is not well followed through in practice. Anghileri (2000) and Askew (2003) each discuss some principles that would seem important in devising appropriate tasks with word problems.

Anghileri (2000) suggests that "teachers can help to develop their pupils' understanding by introducing a wide range of semantic structures and making these the basis for discussion". She also suggests that children should construct word problems to match addition and subtraction number-statements. Askew (2003) discusses the possibility of presenting children with carefully chosen pairs of related problems and suggests that exploration of why one may be more easily solved than another could deepen children's insights into the nature of problem solving. More general 'guidance' is provided by Mason and Johnston –Wilder (2006) who discuss the possibilities of opening up tasks with a view to generating more 'asserting' behaviour from learners.

These principles underpin the construction of approaches to working with word problems used in this study. Three types of task (other than solving) were used. These involved:

- Sorting small sets of problem-statements according to some criteria identified by the learner or sometimes the researcher and discussing the rationale behind the choices made;
- Matching a given problem-statement with one from small set of problems (" which of these problems is most like this one") with appropriate discussion of the reasoning behind the choice;

- Writing problem-statements that 'were like' a given base problem-statement;

The following sections discuss the responses of the learners to such tasks and in particular attempt to identify aspects of the tasks that learners seem to be attending to and any possible shifts in this attention.

4.2 Discussion of data from word problem sorting tasks

Tasks that involved the sorting of problem-statements into groups according to criteria chosen by the learners were a new experience for the children in the study. In the normal course of their work in school it is unlikely that they would have been involved in a sorting task of any kind that was as open ended as the tasks used in the study. Work that the children had carried out with Venn and Carroll diagrams usually involved categories that had been identified by the teacher. It is also unlikely that they would have experienced tasks that involved so little initial guidance and instruction in the course of their normal mathematics lessons. Such a novel task might be considered to be particularly challenging for the learners but may provide a useful 'lens' for identifying the focus of their attention. So when presented with a set of problems to sort, according to criteria of their own choice, what did learners do?

4.2.1 Alan

In Alan's first attempt at sorting he was presented with four problems (Figure 4.2.1) which he grouped into two categories.

Figure 4.2.1 February Second Session Sorting Task Alan	
M At a bus stop 17 passengers got on a bus that was already carrying 26 passengers. How many people were on the bus now?	**U** Ann's parents gave her some money for her birthday. She already had £25. Now she has £41. How much money was she given for her birthday?

O	S
At a bus stop some people got on a bus that was already carrying 28 passengers. There are now 41 people on the bus. How many people got on the bus at the bus stop?	Asif's parents gave him £12 for his birthday. He already had £39. How much money does he have now?

One category he identified as 'buses', the other by implication as 'birthdays or pounds'. His spoken response was:

> "I did these two because these are about buses and they haven't got anything about birthdays or pounds"

This comment might reflect the experience that children in the class had gained using Carroll diagrams where categories need to be mutually exclusive. To ensure this teachers often use a 'has property / doesn't have property' approach rather than a "property A / property B" (where A and B are mutually exclusive properties such as odd and even numbers) hence Alan's comment about one of the sets not exhibiting a particular feature. It may be worth considering how Alan's attention in sorting tasks has been influenced by his work on Carroll diagrams although it is unlikely that the data in this study will reveal much about the nature of this influence.

In assigning labels such as 'buses' and 'birthdays' to the problems it is not clear whether Alan is focusing on the context described or just the mere presence of these words in the problem-statement. With these four particular problem-statements Alan's response is perhaps an obvious one. In many similar activities with ITT students very few have delved any more deeply. From a teacher's point of view we would want Alan to look more deeply and identify the underlying structures implied by the problem-statements. Asking him to group the problems in a different way was an attempt to make him look for different sorting criteria. However, on this occasion, he did not suggest any other criteria for sorting.

Having identified the perhaps obvious response of 'buses' and 'birthdays' for the first sort and then not identifying a category for the second sort may suggest that Alan feels there is nothing more to look for in comparing the problems. An 'expert' or more experienced learner may be aware that there is nearly always more depth and detail to access and may thus be more motivated and have developed appropriate skills to delve deeply below the surface. At this stage of the task Alan may feel that he has 'finished'; after all many tasks in mathematics lessons have a tendency to have only one solution and this is what he has presented.

In the second sorting session a month later Alan was presented with the six problem-statements below (Figure 4.2.2) and again asked to sort them into any categories that he felt were appropriate. On this occasion all of the problems were related to the 'same' described context. They were all about being given or receiving money. This selection was made in order to encourage Alan to look beyond the obvious criteria of different described context (e.g.: Buses and birthdays). If there is not an obvious difference in 'context' then "context" may not get in the way of looking for other similarities and differences.

Figure 4.2.2 March Session Sorting Task Alan
Problem G MonInc1 Paul had £27. His parents gave him £16 for his birthday. How much money does he have now?
Problem H MonDec1 Alan had £21 for his birthday. He spent £6 on a model. How much money does he have now?
Problem I MonInc2 Ann had £28. Her parents gave her some money for her birthday. Now she has £41. How much money was she given for her birthday?
Problem J MonDec2 Asif had £41. He gave some of the money to his sister. He had £28 left. How much money did he give to his sister?
Problem K MonInc3 Sue had some money. On her birthday she was given £16. Now she has £31. How much did she have before her birthday?
Problem L MonDec3 Harry had some money. He spent £12 on a birthday present for his sister and has £19 left. How much did he have before buying the present?

The task was identified to Alan as being to sort the problems in any way that he felt appropriate. He responded by identifying two pairs of problems. It could be that six problems were too much to cope with at this stage and that pairing up problems was an appropriate way of making the task more manageable. If so this could be evidence of effective use of his 'natural' powers in simplifying the task in order to gain access to it.

The dialogue that took place during the task is presented below (Figure 4.2.3):

Figure 4.2.3 March Session Sorting Task Transcript of discussion Alan

Alan originally groups H and J and K and I together in pairs

JB So before you tell me let's see which ones are together, so you've got H and J together, yes, and what else together, any of them?

A K and I.

JB Yes. And are these together or

A No.

JB You're not sure.

A That one says...

JB ... you're not sure. So we're looking at H and G now.

A On H it says Alan had 21 and he spent 6 on a model and so how much does he have now? And this one said he had 41 and he gave some money to his sister and had 28 left, how much money did he give to his sister?

JB OK, so you've told me what... are you changing your mind?

A These together.

JB Right, so you've changed and you're putting H and G together. Why are you changing your mind?

A Because I've just found out that one says how much does he have now and that one says how much does he have now. And I think, like Paul had 27 then gave him 16 how much does he have now? And that one is how much does he have now?

JB So you're looking at the words "have now".

A Yes.

JB OK, I like that. Can you tell me about some of the other ones then? K and I?

A	K and I says that he had some money, on her birthday she was given 16 now she has 31, and it says how much did she have before her birthday. This one says Ann had £28, her parents gave her some money for her birthday, now she has 41, so it's like that, her parents gave her some money, how much did she get.

Alan's discussion focused initially on a pairing of problems H and J.

Problem H MonDec1	**Problem J MonDec2**
Alan had £21 for his birthday. He spent £6 on a model. How much money does he have now?	Asif had £41. He gave some of the money to his sister. He had £28 left. How much money did he give to his sister?

These problems were both 'decreasing' structures but had different unknowns (difference and subtrahend respectively). However when discussing his actions he changes his mind and pairs up Problems H and G. This pairing involves a 'decreasing' and an 'increasing' structure but in each case the unknown quantity is in the same position (total in Problem G, difference in Problem H). It could be that his attention has been drawn to the similarity in the position of the unknown but in discussion he identifies the words 'have now' as being an important influence on his choice.

Alan also pairs Problems K and I together.

Problem K MonInc3	**Problem I MonInc2**
Sue had some money. On her birthday she was given £16. Now she has £31. How much did she have before her birthday	Ann had £28. Her parents gave her some money for her birthday. Now she has £41. How much money was she given for her birthday?

Each of these describes an 'increasing' structure and it seems that in the pairing of problems K and I attention is focusing on the giving of money ('... so it's like that, her parents gave her some money…'). It is interesting that in this pairing the position of the unknowns is different. It seems that Alan has some awareness of this ("How much before her birthday?" "How much did she get?"), though it was not explicitly identified by him as a factor in his choice.

So in the same task Alan seems to pair two problems having the same position of unknown but different structures (increasing and decreasing) and also two problems with different position of unknown but same structure (both increasing). In retrospect drawing this inconsistency, in some way, to Alan's attention may have been a fruitful way forward. It is interesting that the original pairing of Problems H and J was consistent with the pairings of Problems K and I (same structure, different unknown position). If I had been more aware of this at that moment allowing Alan more time to think about the pairings may have resulted in more changes of mind.

This task seemed to generate more 'activity' on Alan's part than the first sorting task. This may be due to the experience that he has now had with sorting and other tasks but may also be due to the design of this particular task. The second sorting task had a greater number of problem-statements to look at and no obvious difference of context between the problems which possibly encouraged him to look for other features.

Perhaps a significant difference in the nature of Alan's activity when working on this task is that he changed his mind about pairings. This could be seen as evidence of Gentner's (1983) 'structure mapping' in action. In a learning situation a learner is unlikely to quickly and easily see similarities (object and relational) between a base and a target domain. It seems entirely reasonable that attention will be drawn to and away from different features as a learner works on a (sorting/matching) task. In this sense there may never be an end to a sorting task in the way that providing an answer to a problem is often an end point of problem solving. There may always be something 'new' to be noticed. From a teacher's viewpoint (or even a researcher's) providing opportunity for a change of mind, as has occurred here, could be seen as creating an atmosphere to encourage shifts in attention.

Such an approach would not fit well with the 'standard model' for primary teaching that requires clear objectives to be made available to learners each lesson and that judgments about learning are made against these objectives as a means of mapping progress. This model of teaching doesn't seem to allow the time for children to engage in 'activity' that may result in them shifting their attention to adequately explore an area. In particular it may not allow a learner the time or the necessary activity to make the 'relational shift' (Gentner, 1988) and focus on the relationship between objects rather than the objects themselves.

With respect to word problems the 'relational shift' could be viewed as a learner becoming aware of the underlying structure of a particular category of problem rather than focusing on surface features. This may occur as an 'ah ha' moment resulting from considerable activity on appropriate tasks. The learner may then notice more easily a wide range of features of problems which hither to may have been invisible to them. Having noticed this aspect of addition and subtraction word problems the awareness that other problems may have underlying structures could change the way in which a learner works with a novel situation.

4.2.2 Barry

Barry's first attempt at sorting involved working with the four problems below (Figure 4.2.4). He was asked to sort them in any way he felt appropriate, just as Alan was.

Figure 4.2.4 February Second Session Sorting Task Barry	
BusInc1 **A** There were 27 passengers on a bus. At the bus stop another 14 passengers got on. How many passengers were on the bus now?	**BusInc2** **C** On a bus there were 35 people. At the bus stop some passengers got on. There were now 49 passengers on the bus. How many people got on the bus at the bus stop?
MonInc1 **G** Paul had £27. His parents gave him £16 for his birthday. How much money does he have now?	**MonInc2** **I** Ann had £28. Her parents gave her some money for her birthday. Now she has £41. How much money was she given for her birthday?

He groups Problems A and G together and suggests that he has done this because each problem-statement contains the number 27.

"Because there's 27… 27 pounds and 27 passengers on the bus"

He also links Problems I and C again justifying his choice with a focus on the numbers that are in the problem-statements.

> "There's 28 and 35... There's 40 and 40 and 30 ... There's 40 and 40 and that's it."

This theme continues in the second sorting session where he is presented with the six problems below (Figure 4.2.5) which are based on a similar context. However the links he makes between the numbers become less obvious for an observer to see as the subsequent dialogue indicates.

Figure 4.2.5 March Session Sorting task Barry	
BusInc1 **A** There were 27 passengers on a bus. At the bus stop another 14 passengers got on. How many passengers were on the bus now?	**BusDec1** **B** On a bus there were 32 passengers. At the bus stop 13 passengers got off. How many passengers were on the bus now?
BusInc2 **C** On a bus there were 35 people. At the bus stop some passengers got on. There were now 49 passengers on the bus. How	**BusDec2** **D** There were 34 people on a bus. At the bus stop some people got off. There were 19 passengers left on the bus. How many passengers got off the bus at the bus
BusInc3 **E** There were some passengers on a bus. At the bus stop another 15 passengers got on the bus. Now there are 33 people on the bus.	**BusDec3** **F** On a bus there were some people. At the bus stop 16 passengers got off. Now there were 22 people on the bus. How many passengers were on the bus before

JB OK. So we've got F, B and A.. And you're very particular about that order, why?

B Yes. Because 22 add a 10 equals 32, then take away another number equals that. They're the 20s.

JB So you're saying 22 add 10 makes 32 in B and 32 take away 5 makes 27. That's how you've linked them together. OK. Anything else you want to tell me about why you've put them together?

B Because these are all 10s. 16, 13, 14.

JB OK, because the numbers are all tens. OK. So have you put those together or haven't you looked at those yet?

B I put these together because this one...

JB So that's E, D and F, sorry E, D and C you've put together.

B I put it all because there's 33, then it goes 34, then 35.

JB Right, so it's 33, 34 and 35 are the three numbers. Brilliant. OK, that's good. So, and you've not only put them in groups but you've given me an order as well. That's very good.

It is as if Barry has decided that connections between the problem-statements must be to do with the numbers and so he attempts to devise a way of connecting the numbers that he sees. He may even have a hierarchy of connections that he (subconsciously) uses. If problems have the same number in them they are grouped together. If the same number is not apparent Barry seems to look for numbers that are close together, for example numbers in the 'tens' or 'thirties'. If this isn't apparent then he constructs some other 'connection' such as:

"22 add a 10 equals 32, then take away another number equals that. They're the 20s."

Rather like Alan (and my ITT students) Barry may find that looking at different aspects of the problem-statements might be obscured by the initial focus of his attention. Barry's initial focus is on the numbers evident in the problem-statement while Alan's was on 'context'. In an attempt to encourage a shift of focus he is asked to try to sort the problems another way without using numbers. This he thinks about for a couple of minutes before making the following response (Figure 4.2.6):

Figure 4.2.6 March Session Sorting Task Barry	
JB	... Then I want you to have a look see if you can think of a different way or sorting them? OK? It might not be easy, but think of a different way... What have you done this time?
B	All of them together.
JB	You put them all together?
B	Yes.
JB	Why?
B	Because they're all about a bus and passengers.
JB	OK, that's a good idea.
B	It's about, and it's about adding passengers and taking away passengers.

This response of sorting the whole set into one group is very unusual in my experience. Not one of the hundreds of ITT students (or any of the children) that I have worked with on sorting tasks have produced such a response even though many of the situations that I have presented them with have had the potential for such a sort. Perhaps it is one of the hidden assumptions of the mathematics classroom that if a task involves sorting a set then at least two subsets need to be generated. Perhaps Barry doesn't comply with these assumptions to quite the extent that other learners seem to. What does seem clear is that Barry responds to tasks in unusual ways. This may be in some way connected with the unstable behaviour that was evident from his work on solving problems.

As Barry has identified that the problems are about "adding passengers and taking away passengers" a suggestion is made that he sorts the problems into "ones that are adding passengers and the ones that are taking away passengers". He works on this and identifies Problems B, D and F as problems about people 'getting off' buses (Figure 4.2.7)

Figure 4.2.7 Barry	**March Session Sorting task**
B	B, D and F are people that are getting off.
JB	So B, D and F are
B	Passengers getting off because they're got off, got off, got off. Triple got offs.
JB	Triple got offs.
B	Look got off, got off, got off.
JB	Got off, got off, got off, so it's a triple got off, I like that. And so let me guess, these are triple ...
B	These are only, there's one got on and double on off, on off. Passengers on off, passengers on off.

Whether it is just the presence of the words "got off" or the meaning of the statement that Barry is attending to is not clear but it does seem to be the case that he has now shifted from attending to the numbers in the statement and is focusing on other aspects. One incident might not be considered to be strong evidence of a real shift of attention particularly as there have been a number of prompts in the session that may have caused this response. However it is interesting to note that in a follow-up session (Figure 4.2.8), three months after any other experience of sorting or matching tasks, that Barry responds differently to sorting the same problems he was presented with in his very first sorting task.

Figure 4.2.8 October Second Session Sorting Task Barry	
From follow-up session in October	
Barry suggests the following grouping… "I got these.."	
G, I	A, C
A	These don't go
B	Yes they do.
B	(speaking quickly) Paul has £27. Ann has £28. His parents gave him £16 for his birthday. Her parents gave her some money for her birthday, How much money does he have now? There are now 49 on the bus. How much money was she given for her birthday?

In this later task Barry appears to be matching up parts of each problem that he sees as connected. The matching seems to be first sentence with first sentence, second sentence with second sentence and so on. This could be due to the order of the sentences or it might be because of the similar structure of the sentences (Paul has… Ann has…; His parents…Her parents…etc). He appears to be working with the immediate words on paper context rather than the imagined context described by the words on paper. Surprisingly he doesn't suggest the contexts of money or buses as reasons for grouping them. Nor does he appear to try to use numbers as criteria for grouping. Whether due to the experience of sorting and matching tasks or some other reason there does seem to have been a change in Barry's focus of attention over the course of the study.

4.2.3 Karen

Karen worked conscientiously on sorting tasks but did not say very much when asked about the reasons for her particular choices. In the first sorting session (Figure 4.2.8) she was presented with the following problems and responded in the way indicated:

Figure 4.2.8 February Second Session Sorting Problems Karen

Karen was given these problems to sort:

BusInc1 ***A*** There were 27 passengers on a bus. At the bus stop another 14 passengers got on. How many passengers were on the bus now?	**MonInc1** **G** Paul had £27. His parents gave him £16 for his birthday. How much money does he have now?
MonInc2 **I** Ann had £28. Her parents gave her some money for her birthday. Now she has £41. How much money was she given for her birthday?	**BusInc2** **C** On a bus there were 35 people. At the bus stop some passengers got on. There were now 49 passengers on the bus. How many people got on the bus at the bus stop?

Karen grouped her problems in a way that seemed to use context (Bus and Money) as criteria.

MonInc1 MonInc2	BusInc1 BusInc2

She did not give any reason for this or offer any other comment.
When asked to sort for a second time according to a different criteria Karen put BusInc1 and MonInc2 together. She found it difficult to articulate her thinking. Her only comment after about 30 seconds of silence was " ... 7 and 7 is 14 and 28 is...".

She sorted these problems into two groups according to what appeared to be the described context although she did not offer any reason or other comment about her choice. On request she attempted to sort the problems again and after a period of silent thought produced a new arrangement along with a comment that appeared to be focusing on the numbers in the problems.

"... 7 and 7 is 14 and 28..."

In the absence of any other obvious (to her) connection she may have had some awareness of the connections between these numbers and used this as the basis for her sort.

A month later in her next experience of a sorting task she identified that all six problems involved money in some way. She also volunteered that some of the problems were "to do with birthdays" but did not suggest any other criteria by which they could be grouped. It is not clear whether Karen did not see any other aspect that she could use or if she was apprehensive about making any type of response. Tasks such as the ones in the study are very different to the normal experience of children in this school. This difference could very easily create some uncertainty in any learner about what was required by such an open-ended task. Karen also experienced some 'matching tasks'. These might be seen as being a little clearer in terms of an outcome that is required. Karen's activity with such tasks is discussed in section 4.3.

4.2.4 Haleema

Haleema's initial response to sorting tasks (Figure 4.2.10) seemed to be to focus on the context described by the problem-statement. This is evident in the notes from the first sorting session. However it is interesting that she chose to mention aspects that were not directly related to her decision. The comment:

> "This carried 17 first then 26 ...no it carried 26 then it carried 17. This one carried 28 then 41"

seemed to suggest a careful reading of the whole problem and is in contrast to her activity on solving tasks where her attention focused on short phrases rather than the whole problem. It could be conjectured that something about the activity promoted by the task has initiated this shift in focus. This suggestion will be discussed in more detail later in the section.

Figure 4.2.10 February Second Session sorting Haleema

Problem M BusInc1a	**Problem O BusInc2a**
At a bus stop 17 passengers got on a bus that was already carrying 26 passengers. How many people were on the bus now?	At a bus stop some people got on a bus that was already carrying 28 passengers. There are now 41 people on the bus. How many people got on the bus at the bus stop?
Problem U MonInc2a Ann's parents gave her some money for her birthday. She already had £25. Now she has £41. How much money was she given for her birthday?	**Problem S MonInc1a** Asif's parents gave him £12 for his birthday. He already had £39. How much money does he have now?

Haleema's first response appeared to be to group her problems according to context:

MonInc1a MonInc2a	BusInc1a BusInc2a

She justified this by saying (pointing to the money problems):

> "both have birthdays, both have money that their parents give them and both say how much money do they have now".

She indicated that the others were about "buses and passengers" and volunteered:

> "This carried 17 first then 26 ...no it carried 26 then it carried 17. This one carried 28 then 41"

In her second attempt Haleema grouped problems together that were 'different':

MonInc1a BusInc2a	MonInc2a BusInc1a

However she did point out that the two subsets that she produced were the same in the sense that:

> "that's about passengers and birthdays (pointing to MonInc1a & BusInc2a) and that's about passengers and birthdays too (pointing to the other two problems)"

Another interesting feature of Haleema's response is the final sort where she produces two groups that are the 'same' in the sense that each contains one 'passenger' and one 'birthday' problem. This was an unexpected response but one that suggests that the task has initiated much thought and possibly creativity on Haleema's part.

In the second sorting session a set of problems based on the 'same' context is used (Figure 4.2.11). Again the intention was to encourage Haleema to focus on some aspect other than context.

Figure 4.2.11 March Session Sorting task Haleema	
Problem A BusInc1 There were 27 passengers on a bus. At the bus stop another 14 passengers got on. How many passengers were on the bus now?	**Problem B BusDec1** On a bus there were 32 passengers. At the bus stop 13 passengers got off. How many passengers were on the bus now?
Problem C BusInc2 On a bus there were 35 people. At the bus stop some passengers got on. There were now 49 passengers on the bus. How many people got on the bus at the bus stop?	**Problem D BusDec2** There were 34 people on a bus. At the bus stop some people got off. There were 19 passengers left on the bus. How many passengers got off the bus at the bus stop?
Problem E BusInc3 There were some passengers on a bus. At the bus stop another 15 passengers got on the bus. Now there are 33 people on the bus. How many passengers were on the bus before it stopped?	**Problem F BusDec3** On a bus there were some people. At the bus stop 16 passengers got off. Now there were 22 people on the bus. How many passengers were on the bus before it stopped?
H There's because, there's 27 there and 35, there's 14 there and there's nothing there and it says 49 there, and it said nothing there.	

JB	Oh right, I'm interested in the nothing, when you said it says nothing there,
H	That's because I said it because they don't say anything and you have to figure it out yourself.

Haleema reads through the problems and immediately states that the problem-statements are about:

> "some going on the bus and some getting off the bus".

In particular she groups A and C together and E, B, F and D together. This isn't quite consistent with her stated criteria but the comments that she makes are perhaps of much more value than the correctness of her sorting. In telling me about why she has grouped A and C together she makes explicit reference to an unknown value that needs to be, in her words, 'figured out'. Such explicit awareness of the idea of an 'unknown' value and its relationship to the other quantities mentioned in the problem-statement may be seen as a development in her algebraic thinking.

Subsequent tasks that Haleema experienced were 'matching' rather than sorting tasks. In looking back on these experiences of sorting it is evident that Haleema has engaged with the tasks in a deeper and more analytical way than she did with the 'straight forward' problem solving.

4.2.5 Further discussion

From the data collected on this small group of learners (and my wider experience) it seems that there are initially two features of word problems that learners tend to focus on when identifying criteria to sort sets of problems. Perhaps the most common is the context that is described by the problem-statement. For example Alan identifies sets he sorts as being "about buses' and about 'birthdays' while Karen's initial sort implies that described context is the criteria she used. The other common focus of attention seems to be the numbers that are present in the problem-statement. For example Barry groups two problems together because they each contain the number 27.

In using either the numbers in a problem-statement, or the context described by it as the criteria for sorting, the children again seem to be using metonymic reasoning. Some individual feature of the problem-statement is being used to represent the whole problem. The features that the children are particularly drawn to might be exerting a 'prototype' effect' on the learners. That is some features seem to be more 'attractive' than others. In the solving tasks, numbers seemed to draw the attention of some children. Subsequently this feature was used as criteria for sorting. However the context described by problem-statements also drew attention initially. There was little reference to this aspect by the children when working on the solving tasks so perhaps there is something about the nature of sorting tasks that draws learners to consider context. One of Gerofsky's (1996) criticisms of the use of word problems is that there is little need for learners to take much notice of the 'set up' component of a word problem. Indeed other authors, such as Sowder (1988), have noted that some children use strategies that tend to ignore the 'set up'. Changing the nature of the task, from solving to sorting, may have had the effect, for some children, of drawing attention towards aspects such as context. Tasks of this sort may be consistent with the 'new ways of working with word problems' suggested by Gerofsky.

A general conjecture arising out of this is the idea that the features that learners attend to in a metonymic way are not only dependent on the nature of the particular structure under consideration but also are dependent on the nature of the task that the learner is set. A non-mathematical example of this might be a driver referring to his 'wheels' but a mechanic referring to his 'motor'. Thus changing a learner's relationship to a structure may make different features more obvious to them. Working on the same object in a different way may cause features which were originally 'stressed' to become 'ignored' and features that were 'ignored' to become 'stressed'. Specifically sorting and matching word problems may cause learners to notice features to which solving tasks did not draw their attention.

It seems that the feature of the problems that learners are first drawn to exerts an influence on their subsequent noticing. In their first experience of sorting Alan and Karen do not produce a second sort when asked to attempt to sort according to a different category, suggesting that they cannot "see" any other feature. In subsequent sorts Barry seems to cling to looking for relationships between the numbers in different problems. It is as if the first focus of attention is so dominant that it obscures attempts to look further or deeper. If attention is drawn to particularly dominant features such as numbers and context then it is drawn away from other aspects. In terms of Mason's structure of awareness, numbers and context are being *stressed* by the learners and other features *ignored,* hence there may be a persistence to continue to be drawn to the stressed features. As Zandieh and Knapp (2006) have pointed out, when a part of a structure is used to represent a whole in a metonymic way then attention is drawn away from other parts. This persistence of attention does seem to be the case within a single session although there is some evidence to suggest that attention is drawn to other features in subsequent sessions. This change may be due to one or a combination of factors:

- the particular selection of problem-statements (in a session) which may have been chosen with the intention of making a feature such as context less obvious as a potential criterion.
- the learners having become more experienced at sorting and matching tasks.
- the gap between sessions allowing the learners to 'forget' the initial dominating feature.
- the children will have experienced many other tasks between sessions which may have had some effect on their tendency to be drawn to particular foci.

Whatever the reason, in the second sorting session Alan seems to be focusing on more detailed aspects of the problems, for example he explicitly identifies the presence of particular words as a possible criterion. He also seems to display richer activity in response to the task. In particular he provides longer and more detailed comment on his reasoning. He also changes his mind. He does this while reading and re-reading problems, and parts of problems, focusing from one to another based on his judgment at the time. This type of activity is in contrast to that typically encouraged by the school which tends to be based on rules and procedures identified by teachers. Alan's behaviour in this session would clearly be classified as 'asserting' and although he may have a natural tendency to work in this way the nature of the sorting task could have acted as an agent in enhancing this behaviour. Mason (2004) has explained that attention shifts rapidly in focus and form, and that this is required in order for appropriate engagement in mathematical thinking. The previous description of Alan's behaviour and the earlier description of Barry working in the follow-up session seems consistent with this view of attention. When attempting to solve problems Alan seemed to be responding to differences in problem structure without being explicit aware of these distinctions. It was suggested that this was consistent with *awareness-in-action*. The nature of the activity that has been provoked by exposure to sorting tasks may be such as to cause him to develop more explicit awareness of these distinctions, thus potentially invoking a shift from *awareness-in-action* to *awareness-in-discipline*.

Furthermore Alan and Barry seem to be working in a way that suggests activity more analytical than metonymic reasoning. After some exposure to sorting tasks they seem to be attempting to look beyond initially prominent features of problem-statements opening themselves up to the possibility of noticing other more useful features of the problem. They seem to be comparing the problem-statements part-by-part looking for similar features. In the language of metaphorical reasoning they seem to be trying to find a mapping between the two problems. In these examples it seems that the mapping is likely to be between objects rather than relationships. The latter might ultimately be desirable and perhaps with continued appropriate exposure to sorting tasks the asserting behaviour provoked may be more likely to achieve this than by sole exposure to solving tasks.

Barry's initial focus seems to be on numbers rather than context but just as Alan's focus seemed to change over time so did Barry's. The aspect of the problems that he seemed to be drawn to were phrases describing actions. In particular on one occasion he identified 'getting off' as a common feature for grouping three problems about buses. Although Alan also came to focus on words his attention tended to be drawn to 'possession' as the words 'have now' featured prominently in his comments. It is not clear whether the particular sets of problems had the effect of drawing attention to 'possession' or 'action' phrases or if there is something about each boy's disposition that drew them to the respective phrases. Perhaps an appropriately structured future study might throw further light on this issue.

So there seems to be some suggestion that providing a sequence of appropriately structured sorting tasks over a period of time might effect shifts in the focus, form and level of a learner's attention and also may provide a stimulus for eliciting asserting behaviour in learners. However in Haleema's case exposure to sorting tasks elicited significant incidents right from the very first task. For example, she provided evidence of attending to the whole problem-statement rather than just key phrases, she produced an unusual rationale for sorting and perhaps most significantly she explicitly identified the 'unknown' variable in problems referring to it as a number that you had to 'figure out'. Perhaps a disappointing aspect of Haleema's performance was that this 'spark' of thoughtful activity disappeared when working on 'straight forward' problem solving later in the study after this experience of sorting.

A theme emerging from the data on sorting tasks that has not yet been explicitly discussed is the tendency for the children to group sets of problem-statements into two subsets. The exception to this was Barry who on one occasion identified a single group. It might be that in time and with appropriately structured sets of problem-statements opportunities to identify a greater number of sets might emerge. It may also be the case that the children's prior experience of sorting was dominated by tasks that involved putting items into one of two identified groups. Lessons that I have observed in the school involving sorting never seemed to require more than this. As a result children may have become attuned to see sorting as involving only two groups. In retrospect consideration of this experience may have been useful when constructing sets of problems for this study.

Beyond Metonymy

Metonymic reasoning is clearly a significant and useful aspect of cognition and behaviour. It has already been suggested that it may play a role in the intuitive, fast, automatic, unconscious and effortless mode of System 1 processing. Leron and Hazzan (2009) also describe S1 processes as inflexible. This seems to be supported by the responses of children in this study who tend to continue to respond to sorting and solving situations in similar ways (focusing on context or numbers for example). In the case of problem-solving tasks metonymic S1 type responses to numbers and words seem inappropriate and draw attention away from more useful features. At a general level there might be two, intertwined, ways forward. One way is to develop tasks that sensitize learners to become aware of and act metonymically in response to other (deeper) aspects of problem-statements. So instead of using keywords and numerals as signifiers for action learners may use some other feature. This seemed to occur in the sorting tasks as some learners seemed to be drawn to the context, a feature that drew no comment in the solving tasks. Changing the nature of the way which the learners worked with word problems appeared to have an effect on the affordances of the task and so, in relation to the attunements of the learners, seemed to produce a new signifier for action, the context of the problem. Subsequent sorting tasks involved explicit and implicit constraints designed to make other signifiers more apparent to learners. Explicitly learners were asked to sort according to different criteria, not numbers for example. Implicitly the design and choice of problem-statements was varied in such a way as to stress some features but attempt to rule out others as possible signifiers. This seemed to result in learners becoming aware of different features but also seemed to prompt the use of a different form of reasoning that overlaid rather than replaced metonymic reasoning. This was metaphorical reasoning, the development of which is the second aspect I suggest to draw on in breaking free of the constraints of S1 reasoning.

An outline of metaphoric reasoning was presented in an earlier chapter and references to it have already been made in analysis of the children's responses to sorting. Barry was earlier described as engaging in activity that involved matching up parts of problems that he saw as connected... "Paul has £27. Ann has £27, His parents gave him, her parents gave her....etc". In the language of metaphoric reasoning he seems to be mapping common features of the problems thus identifying the *ground* between them. This differs from metonymic reasoning in a number of ways. Firstly, Barry is not relying on just one feature or signifier to inform his thinking, he seems to be attempting to take account of a range of the features of the problem. Thus he is *not* using one aspect to represent the whole. His thinking and actions are not unconscious, quick and effortless but are slower and more conscious rather like the description Leron gives of S2 reasoning. As a result of this observation of Barry it is reasonable to suggest that sorting tasks provide particular affordances which along with the attunements of learners may signify metaphorical type reasoning as appropriate action.

It is not my intention to imply that metaphoric reasoning is in any way better than metonymic reasoning or that it does or should replace metonymy in a learner's thinking. Rather the descriptions of each of these forms are attempts to describe how human beings function and each way of functioning is necessary for us to cope with the world. Knowledge and awareness of these ways of functioning would be of great use to learners and their teachers. For example if I have this awareness and I find myself responding in a metonymic way to a situation, if appropriate I may be able to suppress this initial reaction and take a more consciously thought out way forward. This may be an example of *awareness-in-discipline* and the suppression of the initial response may be a function of my 'internal monitor'. As a teacher my knowledge of metonymy and metaphor may also develop my *awareness-in-counsel* as it may enable me to understand the actions of my pupils.

With such a small sample and so little experience with the tasks I cannot claim more than to suggest that there may be benefits to using tasks that involve the sorting of problem-statements. These benefits might relate to a specific understanding of mathematical operations (addition and subtraction in this case) or to more general (word) problem solving skills. However Karen's response to the tasks indicates that some caution is necessary when making claims about the pedagogical role of sorting. Her behaviour suggests she did not 'see' features of the problems that she could use as criteria or that she had a reluctance to commit herself. It could be that such open ended sorting at this point was a 'task too far' for Karen. Perhaps a task that involved a little more direction would be more suitable for her. With this in mind the children also experienced matching tasks. Their responses to these will be discussed in the next section.

4.3 Discussion of Data from Matching Tasks

Tasks involving the matching of problem-statements are a development of the sorting tasks. In matching tasks the usual approach is that a single base problem and a small set of target problems are identified. The learner is encouraged to identify the target problem that seems to him or her most like the base problem and to provide some explanation for this choice. Matching tasks are thus a little less open than the sorting tasks but still provided a new experience for children in the study. So how did they respond to this type of task?

4.3.1 Alan

Alan encountered 'matching' tasks on several occasions in the study. In the second February session following work on a solving task and a sorting task he was presented with the four 'bus' problems below and asked to read them (Figure 4.3.1). During the process of reading he volunteered that he had noticed a difference between A and O. This was a spontaneous unprompted response. Subsequently he was asked to consider if any of these problems were similar to problem U.

Figure 4.3.1 February Second Session Matching Alan	
Problem A There were 27 passengers on a bus. At the bus stop another 14 passengers got on. How many passengers were on the bus now?	**Problem O** At a bus stop some people got on a bus that was already carrying 28 passengers. There are now 41 people on the bus. How many people got on the bus at the bus stop?
Problem M At a bus stop 17 passengers got on a bus that was already carrying 26 passengers. How many people were on the bus now?	**Problem C** On a bus there were 35 people. At the bus stop some passengers got on. There were now 49 passengers on the bus. How many people got on the bus at the bus stop?
Problem U Ann's parents gave her some money for her birthday. She already had £25. Now she has £41. How much money was she given for her birthday?	
Alan was then asked if one of these problems was in any way similar to the 'birthday' Problem U. Alan selected Problem O to match U and explained his reasoning: "Because on U it has got 25 pound and the parents give her some money to make 41 and it's like the same in both of them cos its got 28 passengers already then there are now 41. You have got to think of a number to put on... the number couldn't be the same because that's 25 and that's 28."	

That Alan was volunteering information about things that he had spotted may suggest some element of asserting behaviour. This might be a natural aspect of his behaviour but it could be that at this early stage in his introduction to sorting and matching tasks the nature of the task is affecting the type of activity that he is displaying. At the very least the task is allowing him to display such asserting behaviour. Such opportunities may not be afforded by more common classroom tasks that tend to involve 'solving' as the only way of working with problems.

The matched problems, O and U, do have a similar mathematical structure. The addend and total are given in each case while the augend is unknown and needs to be worked out. From his comments it seems that Alan is attempting to 'match up' elements of each of the problem-statements. It is interesting that he refers explicitly to the quantity that needs to be worked out, in his words 'you have to think of a number to put on'. This seems to be his way of expressing the 'figure it out' number that Haleema referred to. His particular choice of words 'a number to put on' may reveal that he sees the augmentation structure of the problem rather like he did when working on solving problem U earlier in the session (discussed in section 4.1.3). As with his earlier work on solving this problem he does not demonstrate any awareness of using the idea that the problem can be worked out using subtraction.

The comment about 'thinking of a number to put on' could be considered to be evidence of 'relational mapping'. This is the idea that in matching the problems a learner is attending to a similarity about the relationship between variables rather than the variables themselves. If this is the case, Alan may be displaying that he has passed through Gentner's (1988) relational shift. He demonstrates that it is the idea that something is being added on that is important (rather than the amount itself) by his comment that the '*number (added on) couldn't be the same*' in each of the problems. One response to one task is clearly not convincing evidence that Alan has made the 'relational shift'. However, there is some suggestion that the focus of his attention is different from that displayed in the sorting tasks. In those tasks he seemed to focus initially on 'context' and subsequently engaged in activity that seemed to involve comparing and contrasting phrases from different problem-statements until he identified problem-statements that seemed in some way similar to him. This might be considered to be 'object mapping'. Arising from the 'matching' task is a hint that Alan's attention is moving towards the relationship between the objects.

In subsequent sorting tasks Alan tended to revert to more explicit object mapping, so is his attention dependent on the particular task he is working on, or is this apparent change in the focus of his attention to some degree unstable? This question will be further discussed in chapter 5.

There are other similarities about the matched problems that should suggest some caution in drawing conclusions about the focus of Alan's attention. Both problems have the number 41 as the total. It is possible that this similarity initially attracted Alan to match the problems rather than the 'relationship' between the quantities; and that the explanation suggesting an awareness of the relationship only occurred to him when being questioned. This is a similar consideration to that explained when accounting for his expressed thinking about Problem U when solving it earlier in the session (section 4.1.3). It would seem that there is often going to be a difficulty deciding whether an explanation identifies what reasoning actually occurred when working on a task or whether it is some post-hoc justification. However the fact that it has been articulated at any stage of the 'interview' is perhaps a positive feature and does imply a capability on Alan's part of using such reasoning. It may also highlight the need for discussion, questioning and articulation as a pedagogical approach that has a role in affecting awareness and shifting attention.

After matching Problem U Alan was then asked which of the problems he would match with G (Figure 4.3.2)

Figure 4.3.2 Second February Matching Task Alan
Problem *G* Paul had £27. His parents gave him £16 for his birthday. How much money does he have now?
He was then given problem G and asked which 'bus' problem was the same or similar to G. "G and M cos it says Paul had 27 and his parents gave him 16 and it says how much money does he have now and that's an add... and it (points to problem) says at the bus stop 17 passengers got on the bus that was already carrying 26 passengers so you have to add 17 on to 26. How many people are on the bus. It is like that one (points to problem).

He matches M which could be considered to have a similar underlying structure (augmentation structure with unknown total) but differs in the order of mention of variables of the problem-statement. He suggests that each problem is 'an add' and perhaps is aware of the chronological difference as he talks about having to add 17 onto 26 (chronological order of events) rather than adding 26 on to 17 (order of mention). However the commutative property of addition and the 'adding on to the larger number' strategy encouraged by the PNS approach to calculation may have influenced this thinking rather than any unpicking of the chronological order of events.

Does matching of problems that have the same structure on two occasions provide evidence that Alan is aware of these structures? His explanations provide hints that he is getting-a-sense of these ideas and may be working towards a more explicit awareness of the structure of the problems.

4.3.2 Barry

Barry's first experience of a matching task also came in the session on 24th February. On this occasion he was presented with the same four 'bus' problems as Alan but was given a different base problem to match. His response to the task seemed less probing than Alan's, as can be seen from notes taken immediately after the session (Figure 4.3.3).

Figure 4.3.3 February Second Session Matching Task Barry	
Problem A There were 27 passengers on a bus. At the bus stop another 14 passengers got on. How many passengers were on the bus now?	**Problem O** At a bus stop some people got on a bus that was already carrying 28 passengers. There are now 41 people on the bus. How many people got on the bus at the bus stop?
Problem M At a bus stop 17 passengers got on a bus that was already carrying 26 passengers. How many people were on the bus now?	**Problem C** On a bus there were 35 people. At the bus stop some passengers got on. There were now 49 passengers on the bus. How many people got on the bus at the bus stop?
Problem S Asif's parents gave him £12 for his birthday. He already had £39. How much money does he have now?	

Barry was presented with four 'bus' problems A O M C and then asked to read them. He was then shown problem S, a 'birthday' problem. In response to being asked if one of the 'bus' problems was similar to the 'birthday' problem Barry chose A as being similar.

When explaining the reasoning for his choice he suggested that the 12 in problem S is made from a 1 and a 2 and that in problem A there is a 1 from the 14 and a 2 from the 27 which can be put together to make 12

He was then given I to match. He again chose problem A. This time explaining that 27 (in problem A) plus 1 makes 28 (problem I) and that swapping the 14 (in A) makes 41 (in I)

JB If I put different numbers in the problems do you think that you would put the same problems together?

B Don't know... if there were kind of similar tens and units...

Barry's response to the task and his comments suggest that his attention is focused on surface details of the problem-statements. In particular he is drawn to the numbers in the problem-statements. As with the sorting tasks he attempts to manipulate the numbers in whatever way he sees as appropriate to develop an apparent connection between them. The approaches that involved decomposing and reconstructing the digits of numbers outlined above seem to be a 'creation' of Barry's rather than reflection of any teacher initiated classroom task.

From a researcher's viewpoint this behaviour and Barry's tendency to persist with it over time and across variations in task structure is interesting and perhaps is revealing about the features of problem-statements that draw his attention. Numbers, or specifically numerals, in the problem-statements tend to dominate Barry's behaviour to the extent that attempts to draw his attention to other features meet with much 'resistance'. It was earlier suggested that if 'context' is the first aspect of a problem-statement that a learner notices then this initial noticing seems to persist. A similar phenomenon seems apparent with Barry and his focus on numerals.

Alan and Barry also experienced matching tasks in the final session. On this occasion they worked together on the same tasks. This allowed for more interaction between them.

4.3.3 Working together on 'matching' tasks- Alan and Barry

The boys were presented with problems S and U, and asked which of them was most like problem C. As the dialogue below indicates Problem U is considered most like problem C (Figure 4.3.4).

Figure 4.3.4 October Session Matching Task Barry and Alan	
Problem S Asif's parents gave him £12 for his birthday. He already had £39. How much money does he have now?	
Problem U Ann's parents gave her some money for her birthday. She already had £25. Now she has £41. How much money was she given for her birthday?	**Problem C** On a bus there were 35 people. At the bus stop some passengers got on. There were now 49 passengers on the bus. How many people got on the bus at the bus stop?

J I want you to tell me which of those problems is most like (problem) C. So you are looking at problem C..there's problem S and there's problem U...you've got to think about which of those is most like problem C.

Some reading and pointing takes place.

A Yeah that one.....U and C

B I say U and C.

J So you are identifying U as being most like C.

B Well not that much but...

J A bit like C.

A Like Ann's parents gave her some money for her birthday

B But that tells you how much is already there.

A It's just in a different order. Look, look ... and the parents gave her

	some money, some passengers got on, it's like adding like giving money and then ...(unclear ...door alarm goes).
J	That's interesting. So you are saying that on the one it's some passengers getting on the bus.
A	And the other one is like giving some money
J	Okay.
A	Giving passengers and giving money.
J	Giving passengers and giving money. So you're saying it doesn't matter whether it is passengers or whether it is money there is something similar about it.
J	Okay. For Problem U could you just write down the number sentence for the problem. What could you write?

Each of the three problem-statements describes an 'increasing' structure. Problem S has a missing total while Problems U and C each have an unknown addend. Although the problems in the 'matched' pair each have an unknown addend it is not clear that this is the basis on which they were grouped. Other similarities between these problems and differences with problem S may have played a role in influencing the decision to 'match' C and U. For example Problem C and Problem U each refer to numbers whose appearance is similar. Each has a number in the forties and each has a number ending with the digit five. These features have attracted Barry's attention in previous sessions and it may be that he again focused on such aspects despite not explicitly mentioning this in discussion. Alan's explanation seems to suggest that he has identified actions referred to by the problem-statements perhaps seeing 'giving money' and 'passengers getting on a bus' as essentially the same operation. It is interesting that he connects 'giving' money with passengers 'getting on' a bus. Looking at this vocabulary as expressed, out of the context of the problems, would suggest that they are inverse operations, 'giving' implies taking away whereas 'getting on' implies addition. However Alan is clear that both situations involve increasing a particular set by an unknown amount to produce a larger set and he even describes the problems as 'giving passengers and giving money'. This suggests that he sees a connection between the role of the bus in the situation in Problem C and the money that Ann has in Problem U.

At this point it is appropriate to take a slight diversion from discussing the children's noticing to discussing mine. In working with the children and examining and accounting-for their responses my attention seems to focus on different aspects of the problem-statements. I see features that previously I was not consciously aware. For example I now notice that although Problems C and U can be represented by the same arithmetic model there are some perhaps important differences in the situations described by the problem-statements. In addition to the 'giving' and 'getting on' distinction are differences between the nature of the agents and objects in the problems. In Problem U Ann's parents are the agents who act on the object of the problem, the money. In problem C the passengers could be considered to be both agents and objects. Having now identified distinctions that I was previously less aware of, I wonder about the role and effect of such distinctions in problems that children work on in mathematics lessons. When selecting or devising problems for children to work with are these distinctions aspects that might usefully be considered? The role of research in developing the awareness of researcher and teacher, and the subsequent use of distinctions arising from this are points which will be further developed in later discussion.

Returning to Alan, an interesting development in the subsequent matching task was a comment from him that suggested a focus on differences between problems as well as similarities. In identifying G as a problem that 'couldn't' be matched with C he explains:

> "because that's telling you like...you know it straightaway 27 add 16 how much does he have now so I can ... 27 add 16 that is just like... on a bus there is 35 people some passengers got on and it's not telling you how many passengers got on."

This is a different strategy to those previously observed. Here Alan seems to be excluding a problem-statement from consideration because of a perceived difference rather than using perceived similarities to match problems. It might be that he doesn't see G as a problem in the sense of Haylock (2006) as the path to solving it is immediately clear to him; this is not the case with C where there is suggestion that some reasoning has to take place. Whatever the specific reason for the exclusion, this strategy could be one that is particularly afforded by matching tasks.

Alan makes a greater contribution to these joint discussions and seems more perceptive in his analysis of the features of the problems. His comments tend to relate to detail whereas Barry makes general comments such as 'they are the same' and "I agree". These comments may result from Barry not seeing the detail that Alan sees but still wanting to make a contribution to the discussion. In previous tasks he did seem to persist in a focus on surface detail. Barry's response may also be due to the social context of the discussion where he may find it more comfortable to listen and agree than to make more assertive suggestions. This is in contrast to his activity on earlier tasks without the inclusion of Alan.

4.3.4 Karen and Haleema

Karen and Haleema worked jointly on matching tasks on several occasions. The account below refers to two matching tasks undertaken in the second session (Figure 4.3.5).

Figure 4.3.5 February Matching
Karen and Haleema

Problems M and O were placed on the table in front of Karen and Haleema. They were asked to read them. They were then presented with Problem A and asked "Is one of those problems similar or the same as Problem A?"

Problem M (BusInc1a)	***Problem O (BusInc2a)***
At a bus stop 17 passengers got on a bus that was already carrying 26 passengers. How many people were on the bus now?	At a bus stop some people got on a bus that was already carrying 28 passengers. There are now 41 people on the bus. How many people got on the bus at the bus stop?

Problem A (BusInc1)
There were 27 passengers on a bus. At the bus stop another 14 passengers got on. How many passengers were on the bus now?

Initial discussion focused on the presence of passengers and buses in the problems. Eventually BusInc1 and BusInc1a were matched. The reasoning focused on the numbers used in the problems.

> "26 and 27 are close together"
> "14 and 17 are together but has just got 3 more numbers to be together"

The other problem was considered to be different as it had a big number in it:

> "that is close (pointing to 28 Problem O) to it but that is a lot of number (pointing to 41)... a big number".

The comments came from Haleema but Karen indicated that she thought about the problems in the same way.

It is apparent in the first matching task that attention focused on the numbers in the problem-statements. Haleema seemed to dominate the discussion suggesting that she had looked at the closeness of the numbers in the matched problems:

> "26 and 27 are close together"

> "14 and 17 are together but has just got 3 more numbers to be together"

She also gave some indication of excluding a problem from consideration because it contained a big number.

> "that is close (pointing to 28 in Problem O) to it but that is a lot of number (pointing to 41)... a big number".

It is interesting to note that this was the first explicit indication of Haleema identifying differences and using these as part of a strategy to exclude problems from matching. Previously when working on sorting tasks reference was made to similarity rather than difference. This response is further evidence to support the earlier suggestion that there may be something about matching tasks that particularly affords the noticing of differences and use of an exclusion strategy.

The problems used in this task were all set in the same context. The next task involved matching a base problem set in a 'money' context to target problems set in a 'bus' context (Figure 4.3.6).

Figure 4.3.6 March Session Matching Task
Karen and Haleema

Matching problems from different contexts

The four bus problems were laid out in a row on the table in front of the girls. Each girl was given a money problem and asked if they could see a bus problem that was similar or the same in some way as the money problem they had been given. Karen was given Problem S and Haleema had Problem G.

Problem O (BusInc2a) At a bus stop some people got on a bus that was already carrying 28 passengers. There are now 41 people on the bus. How many people got on the bus at the bus stop?	**Problem M** (BusInc1a) At a bus stop 17 passengers got on a bus that was already carrying 26 passengers. How many people were on the bus now?
Problem C (BusInc2) On a bus there were 35 people. At the bus stop some passengers got on. There were now 49 passengers on the bus. How many people got on the bus at the bus stop?	**Problem A** (BusInc1) There were 27 passengers on a bus. At the bus stop another 14 passengers got on. How many passengers were on the bus now?
Problem G (MonInc1) Paul had £27. His parents gave him £16 for his birthday. How much money does he have now?	***Problem S** (MonInc1a)* Asif's parents gave him £12 for his birthday. He already had £39. How much money does he have now?

After about 20 seconds Halemma identified BusInc1 as similar to her problem (MonInc1) volunteering the reason that:

> "they're both 27 and they're 16 and 14 but they're still the same because…there's two more to get to 16."

Karen linked BusInc1a with her problem (MonInc1a) because they say "how much and how many". Karen seemed more eager to respond on this occasion.

Each girl made an appropriate choice in that the structure of the target problems reflected the respective base problems in terms of the position of the unknown quantity. The paired problems also 'matched' the order of mention of variables in the problem-statements. However the explanations put forward to justify these choices seemed not to reflect these similarities. Haleema seems to persist in looking for and 'inventing' connections between the numbers in order to justify her pairing of problem-statements. However Karen seems to adopt a less passive approach than before and is eager to give her view. Her attention is on the words of the problem-statements and in particular she focuses on 'how much and how many'. This explanation doesn't seem to justify her choice but it does give some idea of the features of the problem-statements that she is attending to. It is interesting that on this occasion Karen does not follow Haleema's lead and gives her own reasons.

The next exposure that the girls had to matching tasks was in the April session. An account of the task follows (Figure 4.3.7).

Figure 4.3.7 April Session Matching Task
Karen and Haleema

After working on a solving task with Problem C Karen and Haleema are given problems I and G and asked which is most similar to Problem C.

Problem C BusInc2

On a bus there were 35 people. At the bus stop some passengers got on. There were now 49 passengers on the bus. How many people got on the bus at the bus stop?

I MonInc2	**G MonInc1**
Ann had £28. Her parents gave her some money for her birthday. Now she has £41. How much money was she given for her birthday?	Paul had £27. His parents gave him £16 for his birthday. How much money does he have now?

Problem I is identified by each of the girls as being the most similar to Problem C.

The girls are then asked why they have made this selection.

Although Haleema begins the discussion looking at similarity in the numbers (41 and 49 are linked) Karen tends to have more to say in this part of the session. She is keen to suggest a link between 27 and 35 even though this *does not reflect her original selection* of matching problems. It is hard to see why 27 and 35 should indicate any similarity between the problems particularly as 28 is closer to 35 than 27. Haleema suggests that the choice of matching problems is 'not really to do with the numbers'. However she finds it very hard to articulate exactly what it is to do with and identifies that I and G each include the question "How much?". The discussion finishes with the girls changing their minds and concluding that G is the better match.

The focus of the girls' attention seems to change during this session so that each now provides evidence of attending to aspects that the other attended to in the previous session. Karen appears to have a focus on numbers while Haleema has moved away from this focus as evidenced by her comment that it is 'not really to do with numbers'. Haleema is clearly trying to articulate a reason for her choice with a comment about the presence of 'How much' but is not at the point of making her reasoning explicit. She may be at a stage where some form of shift in her attention is taking place but whatever it is that she is seeing is still too 'foggy' to describe clearly.

As a teacher I might be disappointed with the change of mind at the end of the task as this final choice seemed a less good match. However a task of this nature is very different from the typical task found in mathematics classrooms. The value is, perhaps, all in the activity and not in the obtaining of a correct answer. After all what is the correct answer? As Sfard (1997) wrote, any similarity between base and target (or in this case two problems) is in the mind of the learner.

4.3.5 Further discussion

There are some similarities between the way children responded to matching tasks and their reaction to sorting. Particular themes appear to include:

- a tendency to look for similarities, possibly 'inventing' connections between surface details in the problem-statements, particularly numbers;
- a tendency for a particular focus to persist across time (and across tasks) despite the construction of material designed to 'force' a change of focus;

- evidence, though perhaps ambiguous, of some attention being given to relationships between objects in the problems;

Further discussion of these themes will take place in the next chapter, however two other features have become noticeable as a result of considering matching tasks. The first of these are the comments from Alan that explicitly discuss the differences between features of the problem-statements. This is perhaps part of a strategy to exclude problems from being considered as possible 'matches'. There had at no point been any suggestion from me that an exclusion strategy be used. It could be that this is an approach that Alan is familiar with from other contexts but it is worth considering if the nature of the matching task is such that it may elicit an exclusion strategy from learners. In the previous section a suggestion was made that appropriate activity in sorting tasks might provoke learners to notice features that were not apparent to them in solving tasks. The features that learners seemed to be drawn to were similarities between the problem-statements. With matching tasks we still have the same object for learners to work with (problem-statements) but the nature of the task appears to draw the learners to notice differences rather than, or as well as similarities. Just as sorting tasks offered different affordances to those made available by solving tasks, it may be that matching tasks offer these further possibilities for action to learners. Not only are new possibilities offered but matching tasks seem to strongly signify the noticing of differences and exclusion strategies in a way that solving and sorting tasks do not. Matching tasks can be likened to a particular type of quiz show where contestants are offered several possible answers to choose from for each question. Sometimes the contestant knows the correct answer and gives this straight away but more frequently an approach is taken where particular answers that are seen as being incorrect or unlikely are ruled out one by one. On many occasions even if a contestant knows the correct answer they seem to go through a process of identifying why the other answers are incorrect before confirming their final choice. It seems that a task involving the presentation of a small number of possible responses signifies the use of exclusion strategies of this sort. The most common use of tasks like this in education are the multiple-choice questions used in tests and examinations. I assume that exclusion strategies play a role in learners' responses to these tasks. As this type of question is usually evident in 'assessment' rather than in general mathematics lessons it may be that there is scope for developing lesson based tasks of the matching or multiple choice type where there is opportunity to explore and develop awareness of similarities and differences between different options.

At this point I have become aware that my presentation of tasks to the children has nearly always involved explicit or implicit direction to look for similarities between problem-statements. I have explicitly asked children to find a target problem that is most like a particular base problem or I have asked children to sort problems into groups implying that there must be some common features between them. However, in the children's responses to some matching tasks, they seem to have paid explicit attention to differences. Marton (2006) points out that in order to perceive 'sameness' then 'difference' also needs to be discerned. The power to discern, to make distinctions, is argued by Mason (2004) to be central to education and is a natural power that all learners have. For support he draws on Smith (1954) who identifies very young children's reactions to objects as demonstrating discernment of differences. However the development of this power isn't always encouraged by education (Mason, 2004). This is a tendency that is reflected initially in this study by the focus on sameness in the instructions I gave to the children. A change in the focus of the matching task to look for the problem that is the 'odd one out' might be a way of attempting to develop this aspect of a learner's attention. Marton (2006) develops this point by explaining that it is important to systematically manipulate differences in learning situations as in order to notice differences learners have to encounter them. As he puts it:

> "In order to see how something differs, the learner must previously have seen something that it differs from. So seeing one thing effects how the learner sees another thing subsequently. Not because of the sameness between the two, but because of the sameness and the difference"
>
> (Marton, 2006, page 48)

Tasks that involve the sorting and matching of sets of problem-statements that have been carefully constructed to contain appropriate similarity and difference would seem a fruitful basis for providing such exposure. The task alone though, is not likely to be sufficient to provoke the required development of powers of distinction. Learners need to be encouraged to 'work on' the tasks rather than 'through' them. They need to look for a variety of 'sameness and 'difference' and discuss these features rather than to aim to complete a task and move on to the next. The required 'outcome' of the task needs to be considered carefully and may differ from the type of outcome that tends to be encouraged in many classrooms. This can be exemplified by considering the responses of Karen and Haleema in their third matching task. Each girl produced a very appropriate match between base and target problems. This could be seen as the girls providing a required outcome by meeting typical success criteria (making an appropriate match) and so the task ends there. It may be argued though that the really useful part of the task is in the discussion and reasoning that follows. This activity has had an effect on the focus of the girls' attention and on their awareness of this. Their activity also seems to result in them changing their mind about which problem should be paired with the base problem. They end up with a less appropriate pairing than the original one. So what is the required 'outcome' of such a task? Contrary to my initial thoughts the required outcome is not necessarily to produce what I would consider to be an appropriate match. The appropriate outcome I now feel, is evidence of learners engaging in the sort of activity that has been described, including attempts to articulate reasoning, focus on different aspects of the situation and changes of mind. In short the required outcome should be evidence of learners working *on* tasks rather than *through* them. The nature of the activity required when working on such tasks perhaps requires long term immersion rather than short exposure where superficial 'success criteria' are met.

The second point isn't specific to the matching tasks but has become very apparent to me as I have analysed them. It relates not to the children but to the researcher. I am referring to my own awareness. As would be expected I have noticed much about the children and their responses and as the study has proceeded I feel that the focus of my attention on this aspect has become sharper and deeper. However there is another aspect of my own noticing that has become clear to me. This aspect relates to my awareness of the structure and nature of addition and subtraction word problems. I am now seeing detail arising from problem-statements, of which I previously was unaware, the earlier discussion of agents and objects being a particular example. Has this change in my noticing come as a result of the working with the particular tasks, is it to do with the role of a researcher or is there some other factor that has contributed to this effect?

4.4 Discussion of Data from 'problem construction tasks'.

A rationale underlying the use of sorting and matching tasks is that a learner's attention may be drawn to common features of the problem-statements under consideration. The use of a task that involves constructing or writing a problem that is in some way similar to a designated base problem may be another way of exploring and developing the attention of learners. During the study there were several occasions where as part of a session including other tasks learners were asked to construct their own problems. Usually they were asked to write a problem similar to one that they had previously been working with, though sometimes they were asked to write a problem 'to go with' a particular number-statement.

4.4.1 Alan

In the sense that he produced problem-statements that clearly reflected the structures of the base problem or number-statements Alan was very successful with problem construction tasks. This can be seen from the following example (Figure 4.4.1).

Figure 4.4.1 February Second Session Construction Task Alan
Problem M At a bus stop 17 passengers got on a bus that was already carrying 26 passengers. How many people were on the bus now?
Alan selects problem M and is asked to write a problem that is in some way like problem M. He produces: In one class there are 24 people. Another 8 people come in. How many people are in the class? He is asked to explain how they are similar: "Because it says... um.. cos it's like ... it says at the bus stop there are 17 passengers and another 26 come on. It says how many people on the bus. This says there are 24 and 8 people come on and it says how many people are in the class instead of in the bus"

There are a number of strong similarities between the two problem-statements. The problem that Alan produces has the same structure (X + Y = ?) as Problem M. The quantities used are of the same order, though there does not seem to have been an attempt to use the same numbers. The 'context' of the problem has been altered and refers to people in a class rather than passengers on a bus. It can perhaps be inferred from his comments that Alan sees 'passengers' and 'people' as playing corresponding roles in the two problems. More explicitly he identifies 'class' and 'bus' as corresponding. The use of 'class' could refer to a classroom, or be used in the sense of a group of people. If he is thinking of the latter then it is perhaps interesting that he is linking a physical 'container' (bus) to the more abstract idea of a group. However it is likely, because of the phrase ‘come in’ that he is thinking of a classroom. Problem M has a smaller number of people getting on the bus than are on already. This also seems to be the case with Alan’s problem. He may be intentionally constructing his problem in this way but it is likely that he is reflecting a ‘real’ scenario or even problems from the previous session.

One interesting difference between the base problem and Alan’s problem is the order of mention of variables. Problem M has a non-chronological order of mention of variables whereas Alan’s problem-statement has chronological order. This rearrangement of variables may be deliberate but it is perhaps more likely that this detail is not noticed as Alan probably comprehends the initial problem-statement quickly and easily.

Alan’s responses to number-statements could be considered to be equally accurate. As can be seen below (Figure 4.4.2) each problem-statement is a very appropriate reflection of the arithmetic model presented by the respective number-statements.

Figure 4.4.2 Construction Tasks **Alan**
Writing a problem to go with a number-statement: Presented with **33 + ? = 41** Alan writes: There were 33 children in one class some more children came. There are now 41 children how many more came in. Presented with **? – 6 = 25** Alan writes: There are some sweets in a jar a little naughty kid steals 6 there are now 25. how many sweets were there before the little boy stoal six.

The ease with which he produced these responses suggests that the problem construction task used with these particular initial problem and number-statements may not have been challenging enough to prompt his further development. However Alan's work on these tasks might be considered as evidence that may be used to assess his understanding of the structures that he is likely to encounter in addition and subtraction word problems. On the basis of these observations it could be conjectured that to Alan 'similar' problems are ones that have some similarity in underlying structure rather than in surface detail. This was not the case with the other children.

4.4.2 Barry, Haleema and Karen.

Barry's initial attempt at constructing problem-statements resulted in a problem that reflected the base problem very closely (Figure 4.4.3). The order of mention of variables is the same and he also uses the same numbers in essentially the same position as in the base problem.

Figure 4.4.3 Construction Task Barry
Barry was asked to choose a problem from those that he had been working on in the session. He choose problem S **Problem S MonInc1** **Asif's parents gave him £12 for his birthday. He already had £39.** **How much money does he have now?** He was asked to write a problem that was similar to problem S. There was some discussion about what was required which included Barry asking if you have got to change the numbers. He was told that he could choose the numbers to use. In response he wrote: *Josh played with 12 dixies. He had 39 already.* *How many has he got altogether?* When asked for his thinking he said that he 'just did the numbers but changed the meaning'.

His comment suggests that he thought about the described context when writing his new problem but kept the same numbers. The focus on particular numbers as a similar feature of problem-statements is a consistent feature of Barry's thinking across tasks. Barry's response provides evidence that he sees numbers as the stable part of the problem-statement and identifies the described context as a dimension-of-possible-variation. This is in contrast to Alan, who the data suggests, sees numbers and context as dimensions-of-possible-variation. For Alan, the stable feature of his match is the underlying structure and the relationship between the sizes of the numbers presented. As carried out on this occasion the problem construction task seemed to do little to change the focus of Barry's attention. In a future task it may be worth adding a constraint that prevents a learner from using the numbers from the base problem in their constructed problem.

The focus of attention on surface features of problem-statements also seemed evident in Haleema's and Karen's work on constructing problems (Figures 4.4.4 & 4.4.5).

Figure 4.4.4 Construction Task Haleema
Haleema **Base Problem (M)** At a bus stop 17 passengers got on a bus that was already carrying 26 passengers. How many people were on the bus now? ***Response problem:*** *Lura had 27 marbeles her mum bort her 17 marbeles more how many has she got now Altogether Lura has got 44*

Figure 4.4.5 Construction Task Karen
Karen **Base Problem (U):** Ann's parents gave her some money for her birthday. She already had £25. Now she has £41. How much money was she given for her birthday? ***Response problem:*** *Chloe has 48 Sweets. She gave half to her friend how many sweet do they Both have?* Karen's only expressed reason for constructing this particular problem was that the base problem had a '40' number and so did hers.

Haleema produces a response problem that is similar in structure to the base problem. The numbers in the response problem are the same or very close to the numbers in the base problem. This seems to reflect the similarities that she appeared to be drawn to in the sorting and matching tasks. Karen also expresses the presence of particular numbers (in this case a '40' number) as a factor in the construction of her response problem.

4.4.3 Discussion

The children's exposure to problem construction tasks in the study was minimal. They usually occurred at the end of sessions after a much greater emphasis on sorting and matching tasks and tended to be rushed due to children having to return to their normal school timetable. As a result discussion of the children's responses and appropriate follow-up was limited. Hence consideration of the role of this type of task as a research and/or pedagogical tool is under developed in the study. However there are some thoughts arising from analysis of the task that are worthy of consideration.

Perhaps the most obvious feature of the children's behaviour is the tendency, for three of them, to rely on strategies that they have used in sorting and matching tasks. In particular, the use of similar numbers as a feature for identifying similarity between problem-statements seems to be clearly used. As the constructing tasks took place in the same sessions as other types of task this is perhaps not surprising. Unlike sorting and matching, tasks involving the construction of problem-statements are more evident in the primary mathematics curriculum. For example, a common task in a mathematics scheme that I once worked with was to write a 'story' to go with the a particular calculation that had been worked on. At the time, I didn't consider the purpose of such a task as much as I might have, but I presume that the authors of the scheme were expecting a response that reflected the structure implied by the calculation. Looking back, I now wonder what my children saw as appropriate. Did they see an appropriate 'story' as being one that used the same numbers as the calculation? I doubt that they thought very much about the structure implied by the calculation though most of the time they produced appropriate responses. Perhaps, because of my limited awareness at the time, opportunities to enrich the children's experience of the different underlying structures of the four main operations and of different types of (addition and subtraction) problems were missed. For example, consider the exercise below (Figure 4.4.6):

Figure 4.4.6
From Howell, A, Walker, R & Fletcher, H (1980) *Mathematics for School Teacher's Resource Book 1 and 2* London Addison Wesley page 53 2 Complete. 50-20=? 60-20=? 70-20=? 90-50=? 70-70=? 80-20=? 3 Write a story about the last three examples in Exercise 2.

An appropriate response to question 3 would be a problem-statement using the numbers 90 and 50 that described a reduction situation (Haylock, 2006). At the time I would have been happy with this. I now realise that there would have been opportunity to explore partitioning and comparison structures in response to such a prompt. Another possibility that I now see is the use of X - ?= Z and ? - Y= Z number-statements to explore rather than keeping to the calculation structure (X+Y=, X-Y=) predominantly used by the scheme.

Obtaining correct answers to problems is an outcome that we would want children to experience though we would want them to 'get there' by an appropriate strategy that involves developing an understanding of the problem situation. One particular strategy that I remember using myself as a pupil I have subsequently discovered is advocated by Polya (1990). Amongst his repertoire of approaches is the suggestion that identification of a related problem may be a useful starting point. In primary and secondary school I tried putting less 'complex' numbers into a problem to help me to get a sense of what the problem was asking. I might try substituting decimals with integers for example.

During subsequent mathematics study I noticed similarities between situations that helped me along. When working on a problem about finding out how many matches were required for football leagues of varying sizes, I saw and exploited similarities with the frequently used 'Handshakes' problem. As a teacher I have encouraged learners to look for similar situations in 'division' problems to help decide how to interpret a remainder in the context of the particular problem.

For me 'think of a similar problem' has been an effective approach. However this may not be the case for other learners and is likely to be dependent on exactly what it is that a learner sees as being a 'similar' problem and their ability to remember this information. A study by Krutetskii (1976) may throw some light on this. He found that

> "most capable pupils remember the type and the general character of the operations of a problem that they have solved, but they do not remember a problem's specific data or numbers".
>
> (Krutetskii, 1976, page 299)

In contrast he pointed out that

> 'incapable pupils, on the other hand, usually recall specific numerical data or specific facts about a problem."
>
> (Krutetskii, 1976, page 299)

This is supported in later work by Silver (1979) who found that more able students tended to have the ability to grasp the mathematical structure of a problem but that the least capable seemed to see problem relatedness as determined by contextual details. As I look back I think I saw similar problems as ones that had similar structures though I don't think I was explicitly aware of this at the time. If similarity is seen at a surface level then the "think of a similar problem" strategy is unlikely to be very effective. In particular if a learner sees problem-statements with the same or similar numbers as being alike and these problems have different underlying structures then techniques used to help solve one problem are unlikely to help to solve another 'similar' problem. For three of the children in this study 'thinking of a similar problem' would appear to be an unproductive problem solving strategy. Yet this form of generalisation is important throughout a learner's mathematical development. An example of this may be the use of formula in school mathematics. In order to choose the appropriate formula to solve a problem of some kind then some form of classification of the problem needs to take place. The learner ideally needs to be aware of the similarity between the problem they are working on and other structurally similar problems they have experienced.

The work of Krutetskii (1976) and Silver (1979) unsurprisingly suggests different abilities in the most and least capable students. An area worth exploring is whether it is possible to further develop these abilities. Can the least capable students be aided to attend to mathematical structure rather than to specific data or facts? Put into the context of this study would exposure to appropriate tasks cause Barry, Haleema and Karen to shift the focus of their attention appropriately to more useful features of a problem?

It is likely that the students studied by Krutetskii (1976) and Silver (1979) had experienced problem solving in a fairly traditional way being presented with problem-statements and required to find solutions. They may have been given advice on strategies to help them find a solution but it seems unlikely that they would have worked with problems in any other way. Their experience of problems may thus be rather impoverished in the way that Gerofsky (1996) highlights. However Gerofsky may also have something useful to say about the way forward. Put simply she suggests that word problems should be looked at, and used, in innovative ways. The use of sorting, matching and constructing tasks seem worth investigating as such an innovation. However the availability and use of these tasks may not in itself be enough to encourage the desired development. As may be appreciated from the account of my early experience of working with problem construction, the teacher requires appropriate knowledge and awareness to make work on a task successful. This important issue will be discussed in the conclusion to the chapter.

4.5 Review and further discussion.

The chapter has so far separately presented, analysed and discussed data from each of the four types of task used in the study. This section will review the main points arising from this and will further consider them in a more integrated way that separate consideration would not have allowed. Accounts-of phenomena arising from observation of learners working on solving, sorting, matching and constructing tasks will first be presented followed by accounts-for the phenomena. Accounting-for will draw on the range of theoretical frameworks introduced earlier and will attempt to demonstrate relationships between them. Finally implications for teaching and learning arising out of this discussion will be identified.

4.5.1 Identified phenomena

In the previous sections of this chapter a range of phenomena displayed by the learners while working on solving, sorting, matching and construction tasks were identified. One of the most striking was the instability behaviour evident in Barry's activity. On becoming aware of Barry's instability I became sensitised to it and subsequently became aware of instability in the behaviour of the other learners.

Another significant phenomenon was the tendency of learners to persist with a particular strategy or focus across tasks and over time. This presents a paradox in that persistent behaviour would seem to be the opposite of unstable behaviour. However these apparently conflicting behaviours have been observed, even in the same learner. This state of affairs has set up a dissonance (Festinger, 1957) in the researcher which I will attempt to address. Attempts to account-for the paradox will begin in this chapter but will be discussed in more detail in the next chapter which has a major focus on 'changes over time'.

Other interesting phenomena have also been identified, specifically:

- A tendency in some learners to respond to tasks involving the solving of problems in a 'trigger-like' way with addition usually being the response;
- A strong tendency across the range of tasks for learners to attend to surface features of problem-statements. In particular numerals and keywords seemed to draw attention;
- Some evidence that changing the nature of the task, for example from solving to sorting, or from sorting to matching, affected the focus and nature of the learners subsequent activity;
- Some evidence of the use of different calculating strategies according to apparently perceived distinctions in the structure of similar problems;

4.5.2 Accounting- for phenomena

Some attempt has already been made to account-for the identified phenomena by referring to appropriate theoretical frameworks. No one learning theory seems to entirely account for human behaviour though all theories seem to have something useful to say. Considered together, different theories may provide a fuller explanation of behaviour, so the intention with this section is to go a little further and attempt to describe how these frameworks inter-relate and thus jointly how they may account-for the observed phenomena.

Any consideration of this type needs to take account of the psychology of the agent and the psychology of the object as well as the interaction between them. In this case the agent is the learner and the object is the word problem or more precisely the tasks within which word problems are used. So how can the theoretical frameworks already discussed be used to account-for the responses that children give when working on word problem tasks?

When encountering any situation it seems likely that a learner's initial response will be guided by System 1 type 'reasoning' in order to make a quick 'assessment' of the situation. The use of System 1 when initially encountering any situation may be a normal reaction but in the context of the mathematics classroom the culture that sees speed as important may further influence this sort of response. When encountering a (word) problem solving task a learner is likely to be drawn to particular features of the word problem rather than others. This study suggests that surface features such as numbers and keywords are features that particularly draw attention initially. Action is then carried out dependent on the particular nature of the identified features. This process has already been explained as being a kind of metonymic trigger in the sense that one part of the whole word problem is being used to make 'decisions' about actions that are taken. The features that learners are drawn to are not random but result from the 'psychology' of the word problem task and the attunements that the learner brings to the situation. Word problems offer many affordances; there are many features that a learner can be drawn to and much that a learner can do in response to them. However some of these affordances exert a stronger influence than others. For example problems are to be solved rather than left unsolved; numbers tend to stand out in any problem-statement so perhaps draw attention more than other features. The nature of a word problem thus tends to suggest certain actions.

However the learner comes to any task with their own prior experience. This experience attunes them to respond in particular ways and perhaps over-rides the natural influences exerted by the problem. For example:

- some learners come to word problems attuned to notice keywords (rather than numbers).

- Karen's prior experience, as accounted-for by Davis' frames, seemed to have attuned her to respond to word problems with addition;

- Sowder (1988) has explained that experience with the numbers used in typical calculation exercises attunes learners to 'select' operations based on the combination of numbers in a problem-statement. So 8 and 7 in a problem may suggest addition rather than subtraction.

- Some learners may read and re-read the problem-statement very carefully in great detail and consider the information thoroughly before working towards an answer.

These and other ways of responding will have arisen out of previous experience, sometimes resulting from a teacher's attempts to train behaviour but also occurring as the manifestation of some change in awareness. A learner can only draw on those affordances that they perceive. Perceived affordances Norman (2007) describes as signifiers that give clues about how to act. A particular signifier may result in different actions from different individuals due to their attunements. Consider the case of keywords. Some teachers train children to underline keywords in problem-statements and then to use operations that are implied by the words. For example 'more than' may suggest addition and 'less than' suggest takeaway. A learner responding to keywords in this way is performing a very different kind of action to one who has become aware that some words are perhaps more useful than others but considers these more important words within their relationship with the whole problem-statement. The former is responding metonymically while the latter is doing something more by considering the whole problem and might be considered as functioning with System 2 reasoning.

Some of the phenomena observed can be explained by reference to System 1 processes. An account-for Karen's tendency to add and to persist with addition has already been given in detail. Metonymy seems to play a role here as well as in learners' tendency to be drawn to particular surface details (signifiers). The processes described, although producing undesirable results, seem natural and useful. We could accept that this is how things are and that the phenomena displayed by the children in the study are normal and natural and just need to be worked through. However it could be that there is something about the interaction between these (System 1) processes and the nature of the typical mathematics classroom that has caused the identified phenomena to be particularly prominent. Some features of mathematics teaching that may contribute are:

- a lack of discrimination between situations that learners are presented with. This would include Davis' account of a child's early exposure to calculation but also the common use of problem and calculation exercises based on the same operation;

- teachers' attempts to train behaviour by instructing learners what to do in situations rather than creating tasks where learners have the opportunity to think things through for themselves;

- an emphasis on observed outcome rather than internal awareness;

- an emphasis on pace and short-term remediation of perceived difficulties rather than longer-term immersion in tasks;

Leron and Hazzan (2009) wrote of System 1 processes having a tendency to hijack cognition, implying that a learner would need to exert some effort to overcome this. Rather than helping a learner to exert such effort many commonly used classroom tasks and practices may reinforce the use of System 1 reasoning. Mason and Johnston-Wilder (2004) wrote of the 'inner monitor's' role in remembering goals and guiding activity though being separate from it. It is perhaps a role of the 'inner monitor' to prevent System 1 processes from hijacking reasoning and to draw on other more 'effortful' System 2 processes. Mason sees the overall aim of education as the development of the 'internal monitor'. Thus an aim of education may be to provoke a learner to gain more control of the use of System 2 processes.

Many learners have worked with (word) problem solving tasks and appear to have managed to do so in ways consistent with S2 reasoning though there is evidence (Sowder,1988; Verschaffel et al., 2000) to suggest many have not. Accounting-for this difference among learners is beyond the scope of this study.

However of particular interest is the exploration of pedagogical approaches that might develop the use of a learner's 'internal monitor' in order to encourage their use of S2 type processes. Krutetskii (1976) wrote of the different focus of capable and incapable pupils. The capable attended to the type and general character of problems while the incapable focused on specific numerical detail. Is it possible for an incapable pupil to develop into a capable one by a shift in focus from numerical detail to 'general type and character' of problems and subsequent development of S2 processes?

In this study some learners noticed different things about word problems and performed different actions when they worked on them in a different way. Sorting tasks seemed to prompt noticing of context, and matching tasks seemed to develop an awareness of difference in problem-statements. Though the objects (problems) were the same as in the solving tasks the particular constraints of the sorting and matching tasks seemed to elicit different responses. I do not intend to suggest that prompting a learner to notice something different is as easy as just asking them to *sort* or *match* rather than to *solve*. However I do intend to suggest that the nature of activity that sorting and matching tasks afford might, with sufficient exposure, better promote the opportunity for such noticing.

When considering several problems in a sorting or matching task there is perhaps more for a learner to initially 'take in', so they may be less likely to respond to a single feature in a metonymic way. A sorting task seems to afford consideration of all problem-statements and so may prevent an S1 default trigger-like response from occurring. In the study, all of the children displayed slower, more considered responses when working on sorting and matching. As has been described, problems have been read several times, part of one problem has been systematically considered alongside a similar part of another. An interesting feature was the physical manipulation of the cards on which individual problems were written. This allowed problems to be physically positioned in relation to each other sometimes for ease of comparison but sometimes as a form of recording of their response. When children changed their minds they moved the cards to different relative positions. This form of activity, which all children in the study engaged in, would seem to be something that is afforded by a sorting task presented in this way, but is not in any way possible in a solving task. The children's activity in response to sorting tasks seemed more sophisticated than metonymic reasoning and involved attempts to map parts of one problem to parts of another in a manner more akin to metaphorical reasoning.

The open-ended nature of sorting and matching provided opportunities for immersing learners in tasks in a way not usually possible in solving tasks. When a problem is answered that tends to be the end of the task, and attention moves to the next problem. With sorting there is effectively no single correct answer, so when a learner has sorted problems according to one characteristic they can be asked to find another to sort by. Thus immersing them more deeply into exploration of the features of the problem set they are working on. Other interventions such as changing one base problem for another provided possibility for further discrimination between problems and created additional opportunity for exploration. The nature of activity in response to sorting and matching could nearly always be considered to be *asserting*. Any change in the aspects that the children attended to appeared to develop from within them, and their awareness of these features seemed of a different quality to that I have observed in children displaying "trained" attention to features such as keywords.

Mason (2002) has suggested that you cannot make yourself notice something, nor can you make someone else notice something that you wish them to notice, but it is possible to create conditions in which it is possible to increase the possibility of noticing. Analysis of the data from this study seems to suggest that suitably structured sorting and matching tasks provide an opportunity for developing a learner's 'inner monitor' and subsequently educating their awareness. It is not intended to suggest that a particular sorting/matching task will enable a learner to discern a particular feature or even that exposure to such tasks will develop any awarenesses. However it is conjectured that sorting, matching and construction tasks provide opportunities that are not afforded by solving tasks and so could be used in conjunction with traditional tasks in an effort to develop a learner's 'inner monitor' and prompt their use of System 2 processes.

4.5.3 Implications for teaching and learning

It has already been quite strongly implied that exposure to suitably structured sorting, matching and construction tasks (SMC) based on appropriately selected sets of problem-statements would enhance a learner's experience of mathematics. This exposure should be in addition to more traditional solving tasks rather than replacing them. The general benefits of this exposure may be to educate the awareness of the learner. SMC tasks seem to naturally draw learners' attention to different features of word problems and the opportunity for promoting asserting behaviour may make learners more explicitly aware of the feature to which they are responding. There were many examples in the study of children responding to solving tasks in ways that suggested that they were not explicitly aware of what they were doing. The default trigger-like response of Karen and Alan's tendency to use different calculating strategies for slightly different problems are examples of this. In the sense of Mason's *structure of awareness* these children may be exhibiting *awareness-in-action*. There is some suggestion that SMC tasks helped learners to start to become more explicitly aware of the aspects of problems that they were responding to and also of the reasons for the particular choices they made. This may be evidence of some change in the nature of their awareness, perhaps a shift towards *awareness-in-discipline.* Thus SMC tasks may provide a useful role in eliciting changes in the focus of a learner's attention, and their awareness of this focus. Another benefit of SMC tasks is the opportunity for promoting *asserting* behaviour that they provide. There were many examples in the study of children thinking for themselves, taking the initiative and being prepared to take risks in putting forward embryonic conjectures.

It is acknowledged that these positive observations may have been as much to do with the nature of the sessions and my behaviour as a researcher, as they are with the SMC tasks themselves. This point will be further discussed in the final chapter in considering the validity of the study. However as the nature of specific tasks and the behaviour of the teacher or researcher who is monitoring them are aspects that are well intertwined it is worth identifying here some more general implications for teaching and learning arising from consideration of the sessions before making firmer recommendations later.

- Learning, in the sense of developing awareness, takes time. A single task may generate responses that look like learning has taken place but this study provides evidence to suggest that assessment based on short term consideration of tasks may be misleading. Discussion of this is a significant feature of the next chapter.

- Learners seem to benefit from working with the same or similar objects in a number of different ways. Working on word problems in the different ways implied by solving and SMC tasks appeared beneficial. Working on other objects such as calculations and shapes in a variety of different ways may also have value for learners.

- Learners need opportunities to discriminate between the objects that they are working with. They need opportunities to perceive similarity and difference, to carry out actions on the basis of these perceptions and receive feedback about these actions.

The experiences that these points suggest are needed to assist learners to develop beyond the constraints that are implied by Leron's System 1 reasoning, Davis' frames and metonymic triggers. This all has implications for teachers in respect of their subject and pedagogical knowledge as it is they who devise the tasks that learners work on and monitor their progress on them. In general teachers need to develop an awareness of the learner's awareness (*awareness in counsel*) and use appropriate means to provoke learners' awareness of their *awareness-in-action*. In the case of word problems this might require a good understanding of the various features and structures of problems, the typical ways that learners respond to them and the conditions (including the three points above) required for learning to take place. However learning is a complex phenomenon that is rather less straightforward than is sometimes suggested. The next chapter explores this issue when considering the children's development over time.

5.0 Changes in pupil behaviour over time

5.0 Changes in pupil behaviour over time.

The main focus of the previous section was on the way that learners responded to particular SMC tasks. The analysis focused on the children's behaviour in the moment when working on a task or in the short term following exposure to tasks. Arising from the analysis was a suggestion that long term immersion in SMC tasks might be beneficial to learners. In this chapter attention moves to looking for changes in learners' behaviour over time. It cannot be claimed that any identified changes in behaviour are a direct result of work on SMC tasks. There was deep immersion in the tasks within individual sessions but sessions were about a month apart. Thus the children encountered many other experiences which would have had an impact on their development in addition to the sessions. However SMC tasks were part of this general experience and may have enhanced it.

A comparison of each child's performance on corresponding probes carried out in February and July will initially be presented followed by a discussion of incidents observed in other sessions that might provide evidence for changes in behaviour. Themes arising from this analysis will then be discussed and possible implications for teaching and learning will be considered.

5.1.1 Karen

The table below (Table 5.1.1) provides an outline of the responses that Karen gave to the number and problem statements in the February and July probes. Similar tables are presented for each of the other children. (Shaded squares indicate incorrect responses.)

An initial glance at Karen's performance over the two probes indicates an improved performance in the July probe where she provides six out seven correct answers to number statements (compared to 1 out of 7 in February) and four out of seven correct responses to problem statements (compared to 2 out of 7). It is desirable and perhaps expected that such an improved performance might be detected over a several month period. However a closer look at Karen's responses might be more useful in revealing any changes in her behaviour and as outlined in Chapter 4.1.1 such an analysis identified:

- A shift from correct to incorrect response to problem statement D;
- Recording of calculations in February was a common feature of responses to problems but there was no recording of calculations in response to problems in July;
- No accompanying explanation of answers in February but all answers to problems were presented in the form of a short sentence in July;

Table 5.1.1

Karen		February		July	
Number statement	Problem statement	Number statements	Problem statements	Number statements	Problem statements
16 + 25= □	**A Change 1** Shahina has £16. She is given £25 more by her parents. How much does she have now?	41	£31 20+10=30 6+5=11	41	36 She has £36.00
23 + 19 = □	**B Change 1** Before he was given £23 by his parents Leroy had £19. How much does he have now?	32	32 20+10=30 9+3=12	42	42 Leroy has £42
□- 6 = 28	**C Change 6** On Monday morning all the children in Class 2B are present. Six of the children go home ill. There are 28 children left in the class. How many children were present in the morning?	47	Can't do it	34	Don't know
□ – 5 = 27	**D Change 6** There are 27 children left in the class at home time. During the day 5 children had gone home because they were ill. How many children were present at the beginning of the day?	31	32 27 + 5 =32	32	22 There are 22 people where present at beginning
15 + □ =32	**E Combine 2** There are 32 children in Class 3C. There are 15 boys in the class. How many girls are there?	22	47 30 + 10=40 5 + 2 = 7	17	17 There 17 girls in the class.
16 + □ = 31	**F Combine 2** There are 16 boys in a class of 31 children. How many girls are there?	47	Can't not do it	15	15 There are 15 girls in the class
16 + 17 =	**G Nonsense** Class 4D has 16 boys and 17 girls. How old is the teacher?	Can't do it	This is a silly question	32	Don't know

An argument was presented in Chapter 4.1.1 to suggest that the change in answer in problem D might indicate a shift in Karen's awareness from numbers to words. The contextualising of answers in short sentences in July may also be indicative of an increased awareness of the role played by the words in a problem statement.

Behaviour in the intervening sessions also provided indications that Karen's attention had shifted from looking at the numerals in problem statements to looking at, and taking notice of the words. Evidence that such a change may have at least begun to take place can be found in her responses to other tasks:

> In early sorting tasks she referred to the numbers in problem statements as her criteria for sorting but in a later session she gave reasons involving words, for example a particular set of problems were grouped together because they were 'to do with birthdays'. In a later matching task she pairs up problem statements that contain the words 'how much and how many'
>
> (Chapter 4.1.1)

However in a construction task that immediately followed the tasks above Karen constructed a particular problem with '40' in it because the base problem contained a '40 number' and so did hers. So in Karen's case there is evidence to suggest that some sort of change in her awareness is taking place but that this change is not stable across time or tasks. This instability may not be peculiar to Karen as examination of the other learners demonstrates.

5.1.2 Haleema

The table below (Table 5.1.2) provides an outline of the responses that Haleema gave to the number and problem statements in the February and July probes.

With respect to correct answers Haleema shows no improvement between her February and July attempts at the number statement probes (four out of seven correct on each occasion) and her performance on the problems is poorer (3 correct in July compared to 5 in February). A striking feature of her performance is revealed when looking closely at the patterns of correct and incorrect answers to the problem statements. Four of the problems that she answered correctly in February are answered incorrectly in July and two that she provided no answer for in February are answered correctly in the later probe. So Haleema also seems to display a form of instability but from this evidence it is not clear what aspect of Haleema's behaviour may underlie this phenomenon.

Table 5.1.2

Haleema		February		July	
		Number statements	Problem statements	Number statements	Problem statements
16 + 25= □	**A Change 1** Shahina has £16. She is given £25 more by her parents. How much does she have now?	41	41 £16+£25 T 10+20=30 U 6+5=11	41	40 Shahina has £40 mor altogether.
23 + 19 = □	**B Change 1** Before he was given £23 by his parents Leroy had £19. How much does he have now?	42	42	42	55 He has £55 now.
□- 6 = 28	**C Change 6** On Monday morning all the children in Class 2B are present. Six of the children go home ill. There are 28 children left in the class. How many children were present in the morning?	22	*No response*	34	34 In the present there where 34 children
□ – 5 = 27	**D Change 6** There are 27 children left in the class at home time. During the day 5 children had gone home because they were ill. How many children were present at the beginning of the day?	12	I can't do it	33	In the beginning there where 32 children in the class
15 + □ =32	**E Combine 2** There are 32 children in Class 3C. There are 15 boys in the class. How many girls are there?	22	17	18	Hard
16 + □ = 31	**F Combine 2** There are 16 boys in a class of 31 children. How many girls are there?	15	There are 15 girls.	15	There are 15 girls in the classroom.
16 + 17 =	**G Nonsense** Class 4D has 16 boys and 17 girls. How old is the teacher?	33	I can't tell and it is a silly question.	32	I thin the teacher is 33 years old

A more positive change in her responses is, like Karen, a tendency to move towards presenting her answers to problems in the context of a short sentence that provides some form of indication of the content of the problem statement. It is possible that exposure to SMC tasks over the period of the study may have provoked this change but it could be that some experience in her normal mathematics lessons has caused her to provide answers in such a form. If when solving a problem the focus of her attention is explicitly on generating a sentence it may be drawn away from other important features of a problem. This is a possible explanation of Haleema's poorer performance with problems in July. I initially viewed the generation of these sentences as a very positive feature of Haleema's development, however the 'journey' to this point may say more about her progression than the behaviour itself. One possible description of this journey is that in the course of exposure to SMC tasks Haleema's attention has been drawn more towards the meaning implied by the problem statements and in considering this aspect of a problem she has changed her behaviour and written answers in sentences as this perhaps seemed appropriate to her with this new found awareness. Another possibility is that the production of sentences is trained behaviour that has resulted from directions by her teacher. These different accounts-for the changes in her written response suggest very different things about her development and identify how behaviour in the form of answers and writing can be misleading in assessing a child's progression.

Analysis of data from solving tasks suggested that Haleema may have been using a strategy where her choice of operation was made as a result of the presence of particular phrases in the problem statements. In early attempts at sorting Haleema's attention seemed to be on the context described by the problem statement:

"both have birthdays, both have money..."

Subsequent sorting activity seemed to involve the use of phrases relating to actions:

"some are getting on the bus and some getting off the bus"

This subsequently resulted in her explicitly recognising the unknown quantity in the problem statement. Initially she identified this by the comment "it said nothing there". She then labelled it as something "you have to figure (it) out yourself".

Matching tasks experienced alongside the sorting tasks seemed to draw Haleema's attention to the numbers in problem statements. This was a focus that persisted for her across several sessions. However in the final matching task Haleema did suggest that a pairing that she had made was 'not really to do with the numbers' although she did not manage to articulate what the similarity was to do with.

It is clear from this that some change in Haleema's awareness is beginning to take place over the period between the initial and final probes. She seems to be probing beyond the surface features of problems and is perhaps coming to notice the more implicit features. As numbers seem a readily discernible feature of a problem statement it is not surprising that Haleema's attention is initially drawn there. The data suggests that at some point she is then drawn to noticing and using the nouns and verbs that describe objects and actions in the problem statements. Eventually there is some evidence of attention focusing on the relationships between the numbers as implied by the words of the problem statement.

Haleema's initial behaviour in SMC tasks is rather like that of the 'incapable' pupils described by Krutetskii (1976) and by Silver (1979) who tended to attend to

> "specific numerical data or specific facts about a problem".
>
> (Krutetskii, 1976, p299)

Whereas her later behaviour is more consistent with 'more able' students who can:

> "remember the type and the general character of
> the operations of a problem"
>
> (Krutetskii, 1976, p299)

and who have the ability to grasp the mathematical structure of a problem. So is Haleema moving from being an "incapable" to a "capable" pupil or are such classifications linking behaviour and ability misleading. What is clear is that Haleema has certain powers of awareness and that there is evidence of this awareness changing.

At this stage Haleema is probably 'getting-a-sense-of' the deeper features of problem statements rather than having a clear cut 'ah ha! moment'. Fruitful activity does seem to have taken place though. Another feature of the data relating to Haleema is an apparent "to-ing and fro-ing" in the focus of her attention. She seems to start to probe beyond surface detail by identifying the 'figure it out' number at one point only to subsequently be drawn back to looking at surface detail, numbers, at another. This may reflect the nature of shifts in attention, and hence learning, as an unstable process that may require a prolonged period before a stable change can occur.

Could this hypothesised shift in attention be a factor that has in some way contributed towards the inconsistency in her responses to the February and July probes? The nature of this study is such that a strong case for a causal link cannot really be made. However a conjecture worthy of future study is that there may be such a connection. It seems likely that if the nature of awareness is transient and if a learner requires a prolonged period of immersion in appropriate tasks before shifts in attention become stable then there may be a period of instability in the observable outcomes of tasks. So in Haleema's case it is hypothesised that the unsteady shifts in her awareness have been a factor in producing the unstable observable behaviour in the February and July probes.

5.1.3 Alan

The table below (Table 5.1.3) provides an outline of the responses that Alan gave to the number and problem statements in the February and July probes

Table 5.1.3

Alan		February		July	
		Calculation	Problem	Calculation	Problem
16 + 25= □	**A Change 1** Shahina has £16. She is given £25 more by her parents. How much does she have now?	41	41 £16 + 25 =41	41	41 £16+£25=£41 Shahina has £41
23 + 19 = □	**B Change 1** Before he was given £23 by his parents Leroy had £19. How much does he have now?	42	44 £19 + £23 =44	42	42 23+19=41
□- 6 = 28	**C Change 6** On Monday morning all the children in Class 2B are present. Six of the children go home ill. There are 28 children left in the class. How many children were present in the morning?	34	34	34	?
□ – 5 = 27	**D Change 6** There are 27 children left in the class at home time. During the day 5 children had gone home because they were ill. How many children were present at the beginning of the day?	32	22 27-5=22	32	32 27+5=32
15 + □ =32	**E Combine 2** There are 32 children in Class 3C. There are 15 boys in the class. How many girls are there?	17	17	26	17 32-15=17 There are 17 girls in 3C
16 + □ = 31	**F Combine 2** There are 16 boys in a class of 31 children. How many girls are there?	15	15	15	15 16-31=15
16 + 17 =	**G Nonsense** Class 4D has 16 boys and 17 girls. How old is the teacher?	33	I don't know how old the teacher	33	I can't tell.

Alan's performance in the two probes is fairly consistent although he produces one incorrect answer to a number statement in the July probe after answering all items correctly in February. There is an increase in the overall number of correct responses to problem statements. However Alan does display some instability in that he does not provide an answer to problem C in July after answering it correctly in the initial probe. He also seems to show evidence of starting to provide a written context for his answers in the final probe although this change of behaviour is not as marked as for Karen and Haleema.

In the previous chapter a case was made for suggesting that when working on solving tasks Alan responded to some fine distinctions in the structure of problems. However following this in his early experience of sorting he seemed to focus on more general differences such as the described context of problem statements as indicated by comments such as:

> "these are about buses and they haven't got anything about birthdays or pounds"
>
> (Chapter 4.2.1)

As exposure to sorting and matching tasks continued he responded in ways that suggested changes in his attention might be occurring. He begins to compare and contrast phrases from different problem statements for similarity and demonstrates signs of identifying relationships when he discusses the idea that "You have to think of a number to put on...". Arising from later sessions involving the construction of problem statements is the conjecture that Alan may see similar problems as being ones with a similar underlying structure.

So even though there is little evidence in the probes of objective improvement in his performance it does seem that some form of transformation of Alan's noticing has taken place over the period of his exposure to SMC tasks. It may be that for Alan the probes were not challenging enough to reveal significant change. He did score highly in each of the probes. It is interesting to note that, even if this is the case, it seems that working with similar word problems in different ways (SMC tasks) did provide suitable challenge for him. This seems to support the argument developed earlier, that working with the same objects (word problems) in different ways may provoke profitable changes in awareness. A learner like Alan who is competent at solving word problems may be using the type of approach that Gerofsky (1996) implies as being common place but not representing 'real life' problem solving skills. Changing the task from solving problems to sorting, matching or constructing problems seems to provide opportunity to go beyond such an impoverished approach and develop sensitivities to important features explicitly and implicitly contained within problems.

This account of Alan's behaviour on SMC tasks and his performance in the probes may support the conjecture that learning, in the sense of changes in awareness, may take place with little or no indication being evident in responses to typical maths tasks such as exercises of problems to solve. Observed performance may even decline with initial learning rather like a tennis player learning a new stroke who requires some period of practice and adaptation before the new stroke becomes integrated in such a way as to improve performance. An entry on Mathemapedia (NCETM Online a) provides a useful metaphor that may throw light on this phenomenon:

> The metaphor of *phase transition* from physics is often appropriate in learning mathematics: sometimes energy put in is not manifested in observable change for a time (in physics the energy is achieving a change of state from solid to liquid or liquid to gas). In education, teacher attention and learner tasks may not result in improved performance for a time, and then suddenly there can be a major change in performance.
>
> (NCETM, Online b)

This metaphor also seems to mirror the responses of other children in the study and has implications for the way that information from tests and other assessments is used in education.

5.1.4 Barry

The table below (Table 5.1.4) provides an outline of the responses that Barry gave to the number and problem statements in the February and July probes.

Table 5.1.4

Barry		February		July	
Number statement	Problem Statement	Number statement	Problem Statement	Number statement	Problem Statement
16 + 25= □	**A Change 1** Shahina has £16. She is given £25 more by her parents. How much does she have now?	41 *10+20=30* *5+6=11*	41 *10 + 20=30* *6+5=11* *She has got £41*	31	41 *16+25=41*
23 + 19 = □	**B Change 1** Before he was given £23 by his parents Leroy had £19. How much does he have now?	42	42 *10+20=30* *3+9=12* *£42.00*	42	32 *19+23=32*
□- 6 = 28	**C Change 6** On Monday morning all the children in Class 2B are present. Six of the children go home ill. There are 28 children left in the class. How many children were present in the morning?	34	34 *28+6=34*	32	34 *28+6=34*
□ – 5 = 27	**D Change 6** There are 27 children left in the class at home time. During the day 5 children had gone home because they were ill. How many children were present at the beginning of the day?	32	39 *27+5=12* *27+12=39*	32	32 *27+5=32*
15 + □ =32	**E Combine 2** There are 32 children in Class 3C. There are 15 boys in the class. How many girls are there?	17	17 *32-15=17*	26	17 *15+17=32*
16 + □ = 31	**F Combine 2** There are 16 boys in a class of 31 children. How many girls are there?	15	25 *30-10=20* *6-1=5* *20+5=25 girls*	25	15 *16+15=31*
16 + 17 =	**G Nonsense** Class 4D has 16 boys and 17 girls. How old is the teacher?	33	33 *10+10=20* *6+7=13* *33 years old*	33	*"out of order"*

As identified in the discussion of solving tasks the most striking feature of Barry's responses is the instability implied by the pattern of correct and incorrect answers within and between the February and July 'test' probes. Some of the detail of this instability has already been presented, however, in trying to account-for this inconsistency, it would be useful to identify the 'behaviour' that he engaged in prior to generating these responses. The only data relating to this that was collected in these sessions was the recording that he produced. This will now be considered and subsequently attention will turn to examining his behaviour in SMC tasks.

In February Barry produced some form of recording for all seven problem statements but did so only once in relation to the number statements. In this number statement, and in the case of five of the problems, the recording was some form of working out. This usually involved a partitioning approach to the calculation he was doing. In two of the problems he wrote what appeared to be a summary of the calculation that he had carried out. So there seems to be a clear difference in the recording that Barry produced in the two situations. Problem statements seemed to always prompt some recording whereas number statements usually didn't prompt recording.

It is interesting to note some of the similarities and differences identified in Barry's recording.

In the case of the item pairing A (Table 5.1.5) it seems that Barry adopts a very similar calculation strategy in relation to each of the number and problem statements. The answer to the problem is expressed in a brief sentence.

Table 5.1.5			
Number statement	**Problem statement**	**Number statement**	**Problem statement**
A 16 + 25 = ?	**A Change 1** Shahina has £16. She is given £25 more by her parents. How much does she have now?	41 *10+20=30* *5+6=11*	41 *10 + 20=30* *6+5=11* *She has got £41*

In the item pairing B (Table 5.1.6) there is no recording for the number statement but the calculation strategy recorded for the problem is similar to that for Problem A although on this occasion the answer is not presented in a sentence.

Table 5.1.6			
Number statement	**Problem statement**	**Number statement**	**Problem statement**
B 23 + 19 = ?	**B Change 1** Before he was given £23 by his parents Leroy had £19. How much does he have now?	42	42 *10+20=30* *3+9=12* *£42.00*

Problems statements A and B are similar in a number of ways. They are each Change 1 problems that describe augmentation situations in which a quantity is added to an initial quantity with the subsequent total being the unknown. They involve numbers of a similar magnitude and describe a similar context. They differ in the order of mention of events in the problem statements although in this particular pairing this factor may not make one problem more complex than the other from the learner's point of view and may even be overlooked in a quick reading of the problems. So a similar stimulus in the form of problems A and B seems to elicit a similar response to each, this being the same calculation strategy recorded in a very similar way and a correct answer to each problem. This consistency is not always apparent when comparing other pairs of similar problems. Problems E and F (Table 5.1.7) are both Combine 2 problems. They each involve the same described context and numbers of a similar magnitude. As these are combine problems there is no element of chronology to consider so the difference in order of mention of variables ought not to be as significant as the difference in order of mention in Problems A and B.

Table 5.1.7			
Number statement	**Problem statement**	**Number statement**	**Problem statement**
E 15 + ? = 32	**E Combine 2** There are 32 children in Class 3C. There are 15 boys in the class. How many girls are there?	17	17 *32-15=17*
F 16 + ? =31	**F Combine 2** There are 16 boys in a class of 31 children. How many girls are there?	15	25 *30-10=20* *6-1=5* *20+5=25 girls*

Barry responds to these similar items in different ways. The recording with Problem E seems to represent the calculation that he did (32-15=17). He produces the correct answer but there is no indication of the strategy that he used. In response to problem F he records a partitioning strategy that results in an incorrect answer. This seems due to 'taking away the smaller number' when considering the units segment of the partition. There is an observable difference in behaviour as evidenced by the nature of the recording and it seems likely that the calculation strategy adopted in each item is also different. So in Problems E and F a similar stimulus seems to result in different behaviour in response.

In addition to the instability apparent in the answers given to probe items Barry's behaviour seemed to exhibit some of the following features:

- Problem statements tend to elicit recording. Number statements tend not to elicit recording.
- Recording in July was restricted to what seemed a summary of the calculation used to solve the problem. Recording in February sometimes included an indication of 'working out'.
- Consistent responses to *similar* situations were noted on some occasions. The discussion of the February presentation of problems A and B is an example of this.
- Inconsistent responses to *similar* situations were also noted as evidenced by the above discussion of problems E and F in February.

Barry's observed behaviour in terms of correct/incorrect answers and other recording does seem unstable in the probes. Turning now to look at his response to SMC tasks may reveal more about the focus of his attention and 'internal behaviour'.

Material presented in the section on sorting tasks suggests that Barry's initial attempts at sorting indicated that the focus of his attention was drawn to the numbers in the problem statements. With further experience of sorting, other SMC tasks and interaction with me, Barry's responses to subsequent sorting tasks suggested some changes in his focus. The first indication of a shift of focus (in the third session) seemed to involve the mention of words from the problem statements relating to objects (passengers) and to the actions taken by the objects, for example, "passengers getting off". In the final sorting session Barry seemed to be comparing problem statements systematically phrase-by-phrase and uses verbalisations based on this to justify his sorts.

This transition in attention was not unidirectional even when considering tasks of the same nature. In a sorting task in the fifth session he justifies a sorting decision by referring to connections between numbers:

"Look 41 and 28 then there's 25... just add 1 and 41 and 17 just takeaway a bit. That's 14 then 26, 21, 28 you takeaway 10 takeaway 1..."

This was despite evidence as early as in session 3 that his attention was moving towards greater consideration of the words in the problem statements. Then in later sessions he is taking the phrase-by-phrase approach. This "to-ing and fro-ing" of attention is also seen in Barry's responses to different SMC tasks. For example in the final session he constructs a similar problem using the rationale that his problem statement has the same numbers as the base problem.

The focus of Barry's attention does appear to have generally shifted over the course of the study and seems to have done so in a useful way. An analysis of problem statements systematically phrase-by-phrase looking for similarity and difference is perhaps more desirable than linking problems by 'created' connections between the numbers they contain. However as indicated above this change is not stable. In working at different times and on different tasks he sometimes 'reverts' to noticing that is less deep than he has previously demonstrated.

Consideration of data from the SMC tasks presents a more ordered view of Barry's progress than that suggested by his recording and his answers to the number and problem statements in the initial and final probes. The picture presented is one of 'moving forward' in the sense of attending to aspects not previously noticed with 'steps back' where awareness of these aspects is not apparent, followed by further 'steps forward' and so on. The view of Barry's progress suggested by his recording in the probes alone could be viewed as 'stepping back', in the sense that in the second probe he produces incorrect answers to items that he has previously answered correctly. A different view is that the two probes represent small snapshots at different points in time that should not be seen as definitive in identifying 'progress' as a whole but as data that contributes to the picture of Barry's development over time. The image that is becoming apparent is not so much one of 'stepping back' in learning but one that reflects instability which may be part of the normal process of learning.

It seems likely that there is some connection between the nature of Barry's shifts in attention and observable behaviour in the form of answers and recording. It seems likely that whether he is attending to numbers, words, phrases or structures when working to solve a problem statement has an effect on the answer that is given. However the exact nature of the relationship between attention and other 'internal' behaviour; and observable behaviour is not clear.

5.2 Discussion

Considering the responses of the children to working with word problems over time seems to reveal two main themes:

- The presence of various forms of instability in the behaviour of each of the learners;
- The suggestion of shifts in attention over time for each of the learners;

In contrast to each of these themes are observations of the learners that suggest a tendency to persist with a particular behaviour or a particular focus of attention despite exposure to experiences intended prompt appropriate changes. Each of these themes will now be discussed in more detail and attempts to account for the phenomena will be made.

5.2.1 Instability

In being drawn to what I have 'noticed' as instability in the behaviour of the learners I have also become aware of an underlying assumption that I have made. I have assumed that it is reasonable to expect a learner to respond in similar ways to similar situations. Hence when a learner has displayed behaviour that is not consistent with my assumption I have become aware of it. In the early stages of my experience as a teacher I would have viewed unstable behaviour as 'strange' and possibly frustrating, particularly when unstable behaviour seemed to suggest that children were not learning. Now as a more experienced teacher and researcher I see unstable behaviour as an area of interest warranting further exploration. Some attempt at this exploration has already taken place in the accounts-of the particular children especially in the case of Barry. On reflection I am now considering the validity of my assumption. At a general level all of the children have displayed some form of instability in response to what, on the surface, seem similar situations. However it is worth considering whether the situations are as similar as I first thought. Are there small differences in the 'similar' situations that may have the effect of prompting different behaviour?

Examples of such differences might include:

- The ordering of numbers in a problem (Problem E has larger number first while problem F has the smaller number first).
- The inclusion of irrelevant information. (Problem E has the class number while problem F doesn't indicate this information)

Even when the 'same' problem statement with exactly the same words and numbers is presented in the same position in a set of problems, the situation for the learner may not be the same. For example the items that a learner has worked on previously may have an effect on the response to a particular item on different occasions. In both the February and July probes the order of items was the same but experiences prior to each of the probe sessions were different. The July probe was presented after several sessions involving solving, sorting, matching and writing problems. The February probe was not preceded by such experience. These sessions were of course only a small part of the total experience of the children during the period of the study. In school they will have experienced a variety of forms of teacher intervention and many things will have happened in their lives outside of school. Hence the learner may have developed various short-term 'sensitivities' and abilities as a result of the sessions and other experiences. So although the probes remain the 'same' the learner's capabilities and sensitivities are likely to have changed so the situation the learner encounters is not the 'same'. Put another way a learner may bring different attunements to a task on different occasions.

In previous discussion Problems E and F were considered as similar. However Problem E was presented earlier in the probe then problem F so the response to F may have been affected by the learner having experienced problem E first. Problem E followed problem C and Problem F followed D. Again this difference in order of presentation of items makes the situation encountered by the learner at least slightly different, hence creating the possibility of eliciting a different response. In the case of a number of classroom activities, consideration of the effects of immediately prior activities might be significant. A number of situations come to mind:

- A teacher who introduces 2-digit multiplication in the main part of a lesson after a mental and oral starter focusing on multiplication facts is clearly creating a different situation for learners than a teacher who focused on properties of shape prior to introducing 2-digit multiplication.
- Text books that present a set of problems following a set of related calculations create a different situation for the learner than those that present the same problems in relevant isolation.
- A set of calculations that follow a particular pattern (e.g.: 25 +37= ; 2.5 + 3.7=) may sensitise learners in a way that the same calculations presented randomly may not.

Is it reasonable to expect stability in behaviour in response to mathematical tasks when it may be that no two situations are entirely the same? I am now starting to think that such an expectation is not reasonable and may not be helpful when considering effective approaches to teaching and learning. However it is likely that there are other factors that contribute to unstable behaviour. One of these might be the way a learner's focus of attention and level of awareness changes, but before considering this, other examples of unstable behaviour will be considered.

My initial reaction to noticing unstable behaviour was one of surprise although it perhaps should not have been. On an anecdotal level many teachers 'complain' that what their learners seem to be able to do one day they can't do the next.

Several authors (Houssart (2007), Ginsburg (1997)) have also acknowledged and discussed the phenomenon. In particular Ginsburg (1977) points out that:

> The individual child does not always succeed or always fail, and does not always use one single procedure to get these right or wrong answers. Rather the child's answers involve complex mixtures of success and failure, and he uses several different procedures to get his answers.
>
> (Ginsburg, 1977, p 116)

He goes on to provide general-accounts for the typical errors that children make in mathematics but points out that errors often result from "organised strategies and rules". He complains that children often see maths as an activity that is isolated from everyday concerns, a game with its own unique rules that do not apply to other situations. This is consistent with Gerofsky's (1996) discussion of word problems in which she sees some strategies for solving word problems as having little value in any other situations. However Ginsburg does point out that even those children who make many errors do possess unexpected and useful strengths that should be used as the basis for effective learning. Whilst acknowledging the existence of unstable behaviour in pupils Ginsburg's discussion does implicitly seem to view it negatively, but if it is a fairly common phenomenon might it serve some useful purpose?

Unstable behaviour, or cognitive variability, as he refers to it has been examined in more depth by Siegler (2007). He draws on a wide range of evidence to establish the variable nature of children's thinking and discusses the implications of this. In particular he claims that

> "... thought and action are highly variable within individual infants, toddlers, children and adults"
>
> (Siegler, 2007, page104)

and that variability in strategy is not attributable to an

> "... orderly progression from less adequate strategies to somewhat more adequate strategies".
>
> (Siegler, 2007, page 104)

For example, he identifies the use of less adequate strategies by learners in trials after they have already drawn on more advanced ones in previous trials. He does make it clear that in the data he has examined regressions are temporary with the overall trajectory of change being upward. Progress is identified as reflecting "a back and forth competition rather than a forward march". There has been a tendency to see variability as unwanted but Siegler sees it as important, pointing to evidence that greater initial variability of strategy seems to predict greater subsequent learning. In accounting for this observation he draws on several other authors, concluding that some form of instability is necessary for any system to change, as "learning is most likely when previously dominant approaches weaken".

Much of the evidence drawn on by Siegler (2007) and by Ginsburg (1977) tends to be what now would be referred to as 'outcomes'. In practice this would include children obtaining correct or incorrect answers or using this or that overt strategy. The discussion may be further informed by exploring less explicit aspects of their behaviour.

5.2.2 Awareness, attention and instability

Clearly at the level of correct/incorrect answers we would want learners' behaviour to change from incorrect to correct responses over time. We would probably not consider such changes to be unstable behaviour but rather as positive, progressive and desirable. No teacher would normally welcome a movement in the opposite direction. However analysis of this level of response is perhaps a blunt tool for providing evidence of the learning of a child. Several authors (Sowder,1988; Verschaffel, Greer & de Corte, 2000) have already highlighted that correct answers are not necessarily indicators of correct thinking or appropriate understanding. Other less obvious aspects of behaviour have been considered in this study with a view to throwing light on the nature of a learner's activity while working on a task. Analysis of data related to these aspects has suggested changes in the focus of a learner's attention over time. While there seems to be evidence for a general movement from attending to surface details such as numbers or individual words in problem statements towards the meaning implied by the statements and the underlying structure of problems, this movement has not been uni-directional or constant across tasks. Having displayed evidence of attending to more detailed features on one occasion seems to be no guarantee that this awareness will be drawn to such detail on another occasion. It would seem reasonable to assume that this variation in attention is normal and natural. Although I feel inclined to assume that being drawn to more detailed aspects of a situation is positive it is likely that there are benefits to learners in the fluctuation of awareness that I have observed. If we are always drawn to the detail then we may miss more superficial aspects which sometimes might be important. 'We may not be able to see the wood for the trees'. This flexibility of attention that allows for 'retreat' to focus on more readily discernible features may be particularly helpful when learners experience the pressures involved with encountering novel situations

This view of fluctuating attention is rather like a component of Kieren and Pirie's (1994) onion model of understanding. This model can be thought of as eight nested layers or modes that describe the growth of understanding for a particular person on any specific topic. To give a flavour of the nature of the model it may be useful to describe the inner most levels. Understanding a particular topic is seen to begin with *primitive knowing.* This "is what the observer, the teacher or researcher assumes the person doing the understanding can do initially." A complete picture of any learner's initial knowledge in an area is impossible to ascertain but at the beginning of this study it might include assumptions that the children could add and subtract two-digit numbers, were implicitly aware of some underlying structures of the operations and were able to read and understand sentences of the complexity of those in the problem statements used in the study. The second level is identified as *image making* in which a learner is encouraged "to make distinctions in previous knowing and use it in new ways". Sorting of problem statements may encourage the children to become aware of similarities and differences and use them to establish in their minds 'new categories' within the range of possible addition and subtraction problems. The next level is *image* *having*. Here a mental construct can be used without recourse to the activity that brought it into being. This level might be exemplified in the current study by a learner recognising a particular type of problem without having to work through the comparison process involved with sorting. It may be possible to further outline the model and illustrate it in terms of the functioning of children in this study, however, it is not the layers themselves that are particularly useful to the current discussion but some other properties of the model.

Kieren and Pirie (1994) emphasise that in their view the growth of understanding is a "dynamic and organising process" that is *not* mono-directional. They explain that there is some hierarchy in the model but that inner and outer levels of activity are not meant to imply low level and high level mathematics respectively. In fact it is suggested that a learner's outer level activity in one topic will provide the primitive knowing for another topic. The feature of the model that is particularly relevant to the current discussion is the idea that a learner's growth of understanding in relation to the layers is *not* mono-directional. Inherent in this idea is the concept of '*folding back'*. This is the phenomenon of learners who have displayed evidence of working at an outer layer of activity reverting to performing at an inner layer. This 'stepping back' is not considered as negative but as an essential process in the development of understanding. Kieren and Pirie explain that, when faced with a problem that is not immediately solvable, a learner needs to '*fold back'* to a more primitive level in order to extend their current inadequate understanding. However this inner level is now not as it was but has been shaped by activity at the outer level and has thus, using the onion metaphor, become a thicker and firmer layer. So folding back is seen as a normal and entirely necessary feature of the development of understanding. This seems consistent with Siegler's (2007) suggestion that instability is necessary for any system to change.

Fluctuation of a learner's attention over time might also be reflected in a similar model. When presented with problems that are not immediately solvable a learner's focus of attention may shift back towards more readily discernible features in order to get-a-sense of the situation. However previous experience of working with less explicit features will have had a role in developing the subsequent activity with surface details. This would suggest that a learner focusing their attention on numbers in a task after some evidence of having discerned aspects of the mathematical structure of problems is engaging in a different quality of activity than a learner whose attention has apparently not moved beyond numbers. However, data in the study does not really provide any insight into differences in the nature of activity when folding back occurs, nor is it helpful in identifying specifically how a particular layer of understanding is "thicker and firmer". This is not surprising as the study was not initially designed to examine instability and 'folding back'. A suitably structured future study using SMC tasks might reveal more of the nature of shifts of attention in relation to instability and folding back.

Kieren and Pirie suggest encountering 'problems that are not immediately solvable' as the sort of situation in which folding back may occur. Such situations might include variations on a task; for example a learner might display a higher level of sensitivity (perhaps a phrase-by-phrase approach) when working on a sorting task but fold back to a lower level (looking at connections between numbers) when moving to a matching task. A period of time between experiencing tasks of a particular kind may also cause a learner to fold back. It is conjectured that the fluctuation of awareness apparent in the learners in the study is not random and haphazard behaviour, but is a normal, natural and necessary recursive process reflecting that described by Kieren and Pirie.

If folding back is a necessary aspect in the development of attention (and understanding) then an implication is that learners be put in situations where it is necessary. However rather than just finding 'problems that are not immediately solvable' to work with, careful choice of related tasks may be a fruitful way of encouraging folding back and hence subsequent development of awareness. In general, exploiting the idea of dimensions-of-possible variation in task design (Mason and Johnston-Wilder, 2006) might generate useful related tasks. In particular, in the current study, the sorting, matching and construction tasks used are closely related but seem to be different enough to provide a challenge that will encourage folding back. It is particularly striking that Alan, although a very competent problem solver, seems to shift his focus back to surface features when encountering SMC tasks. The variation of task from solving problems to sorting them seems a sufficient stimulus to provoke this form of folding back.

When children in school work on a particular task and meet the success criteria associated with the focused learning objective they are usually expected to move forward to the next objective with its associated targets. Perhaps an approach to be considered is that learners should explore more variations of the task in a way that immerses them in the area of study, prompting multiple moves between different layers of sensitivity, thus informing and shaping their awareness and understanding of the area. Successful solving of problems may not elicit folding back, but exposure to SMC tasks based on problem statements might just provide such a stimulus.

If it is accepted that development in attention is non-unidirectional, with learners attending to different aspects on different occasions in the ways previously described, then there may be some connection between this and the unstable responses demonstrated within and between tasks. Behaviour is likely to be dependent on attention to some extent. Therefore if attention fluctuates it is likely that behaviour will fluctuate. This in turn may affect the production of correct/incorrect answers. However the connections between awareness and behaviour do seem to have looseness to them. It has already been seen in the previous discussion of Alan and the *phase transition metaphor,* that significant and desirable shifts in attention have not been always been associated with the production of correct answers. Thus when looking for evidence of development teachers and researchers need to delve beyond answers.

Initially as a teacher, my reactions to unstable behaviour have been negative, particularly when evident in the form of answers to questions. Subsequently, with experience, I came to see unstable behaviour as a normal and acceptable part of learning. In the light of considering the performance of the children in the study, I have come to see unstable behaviour as part of a phenomenon that is wholly necessary for appropriate learning and development to take place.

I have argued that unstable behaviour may in some way result from natural, normal and necessary fluctuations in awareness. I have also suggested that what seems to be unstable behaviour in response to similar situations may result from small differences between the situations. These differences may be in the presentation of the task itself or may be to do with prior experiences of the learner. If small differences do elicit different responses then it implies that the learner is in some way aware of them. It might then be argued that much of what has been described as unstable behaviour may be the result of fluctuations and shifts in attention. Some of this variation may be the result of folding back but it is possible that some is the result of other factors such as feeling ill, tired or being bored or lacking in motivation.

If it is accepted that fluctuations in attention and unstable behaviour are normal and natural, and even necessary, then there are implications for the way that we teach, but before considering this it may be useful to explore a contrasting theme.

5.2.3 Persistence of behaviour over time and across task

The previous sections discussed observations of the children that suggested evidence of unstable behaviour and fluctuation of their attention. However a contrasting theme that emerged was that of children persisting with a particular focus of attention or behaviour. Two examples from prior discussion may serve to illustrate this feature:

> When presented with a problem to solve, Karen exhibited a strong tendency to proceed by adding the numbers present in the problem statement. She either used addition

or did not produce an answer, in every case except in the last session where there was an indication that she was using subtraction.

Chapter 4.1.1

When working on various SMC tasks throughout the study Barry displayed a tendency to use connections between numbers as the criteria for making decisions. This type of response persisted even when problem statements were constructed that used numbers with no obvious connections. In such cases Barry became quite creative in developing ways of 'linking' the numbers. It was as if, having developed this as a strategy, he needed to carry on using it, even though it was not perhaps always the most appropriate approach.

Chapter 5.1.4

We might expect, even desire and encourage, persistence with particular behaviours when they are ones that we feel are appropriate to the situation in hand. However, in these examples, the particular focus or behaviour tends to persist into situations where they are inappropriate. In these examples and others like them, there seems to be persistence of activity of a type that tends not to be desired whilst at other times there is the suggestion that children have shifted away from more appropriate behaviour. This has been described in the previous sections about unstable behaviour and moving focus. On the surface this seems a confusing state of affairs and may even be disturbing from a teacher's point of view. However, in looking at the observations as a researcher, it seems a phenomenon that needs some accounting-for.

This persistence with a particular type of activity has resonances with behaviour that I have observed in classes that I have taught. I remember a number of instances when pupils have been working on a calculation exercise practising a particular operation and then when an example requiring a different operation has been presented they continue to use the initial operation. For example, when practising subtraction and then being presented with addition calculations some pupils have continued to use subtraction. I have tended to think of this as 'task inertia', the continuance of a particular behaviour that has been established as the result of working on a particular task despite that behaviour no longer being appropriate.

Although not always the case, 'task inertia' has seemed more prevalent when learners have encountered new or novel situations. This is clearly not so in the subtraction/addition example above but might be illustrated by considering the response of some of my pupils when being introduced to subtraction situations that require exchange between tens and units (ones). Children have seemed to adopt this technique fairly quickly but many to my dismay have continued to apply it to situations where exchange is not required!!

Similar behaviour has been recognised by others. For example Hughes, Desforges, Mitchell and Carre (2000) describe an observation of a child in a Japanese classroom. In one lesson the child is finding pairs of numbers that total five and continues to do this in the next lesson even though the task has been altered to finding numbers that total six. These examples could be accounted for as a desire in learners to continue a success pattern and might also be illustrative of learners functioning at the level of awareness-in-action where they may not be explicitly aware of their actions.

Barry's persistence with making connections between increasingly unrelated numbers and Karen's tendency to always add were their first responses to tasks in the sessions with me. It seemed as if whatever features of a situation the learners attended to initially became a focus that is maintained. It is conjectured that there is a tendency for the behaviour that a learner elicits in response to a novel situation to persist over time and across similar tasks even if this behaviour is not the most appropriate response to the situation. This being particularly so if the learner initially meets with success. So if a learner attends to the numbers in problem statements in their first experience of sorting tasks then this focus of attention may persist in work in similar situations and may only shift after prolonged immersion in SMC tasks.

There are some differences between the observations of persistent behaviour in the study and examples from my teaching. In the latter the 'persistent behaviour' has been taught, often modelled, by the teacher whereas in the study the behaviour has been 'generated' by the children in response to the task. It is not clear what effect this difference is likely to have on the persistence of any particular focus or behaviour. However it may be that any response that has been 'generated' by the learner is due to attunements that have resulted from their prior experiences. As these attunements are likely to have developed over a prolonged period, the learner's response may be more resistant to change than any newly taught behaviour which may not have had time to 'become a part of the learner' in the same way.

In Chapter 4.1 an account explaining Karen's tendency to use addition was presented but the account did not *address* the persistence of this behaviour. At the time of the study Karen will have had experience of operations other than addition but seems not to use them in response to problems. Davis (1984), in discussing the *'permanence of productions',* suggests that 'representations are not actually deleted from memory' and implies that 'contradictory information and competing alternative procedures, will come to coexist in our memories'. Perhaps, as a result of Karen's experience, the *undifferentiated binary-operation frame* causes her to use addition and dominates any other possible response she could make. Having had so much more experience of addition than any other operation addition 'wins' any competition for retrieval when a way forward is not obvious.

The development of the *undifferentiated binary-operation frame* returning addition has arguably resulted from considerable experience over a long period of time. It seems unlikely that other more newly added frames will 'compete' in the short term. Hence it seems that, in order to develop Karen's performance with problem statements, prolonged immersion in tasks that are likely to develop other more appropriate frames is required. There is just a hint in the final session of the study that Karen is becoming aware of subtraction as a possible response to problems. This may have in part been due to her work on SMC tasks but may also be due to gaining more experience of subtraction in her general school maths lessons. It does seem clear though, that her use of addition as a response to problem statements was a firmly established behaviour, and that a prolonged period of activity was required before signs of any shift from the behaviour became apparent.

Whereas Karen's behaviour seems consistent with the description that Davis gives of the specific *undifferentiated binary-operation frame,* Barry's behaviour in SMC tasks might be explained by more general principles. Davis describes how, in human information processing, 'discriminations are only as fine as necessary'. During Barry's first attempt at sorting he is drawn, like many learners, to using a surface feature of the problem statements as a criterion for sorting. In particular he uses connections that he sees between the numbers present. As the numbers present are noticeably different from words, attention may be drawn to this feature initially. If it seems possible to discriminate using these numbers then no further examination of the problem statements is necessary. The internal behaviour underlying this first response to sorting will not, in Davis' terms, be 'deleted' and so 'numbers' will continue to compete with other features in future SMC tasks. The more that Barry is drawn to numbers the harder it will become for other, perhaps more desirable features, to compete for his attention. His behaviour may also be interpreted by the 'highest program must run principle'. If no other feature becomes obvious to him then he will use 'numbers' as his best option.

It may be that the there is a spiral of activity that converges attention towards the feature that is first noticed. A learner will in the early stages of experience with a particular task perhaps be drawn to a particular feature. Hence the learner is being provoked into using metonymic reasoning. The feature that they are drawn to may exhibit prototype characteristics and in doing so draw attention away from other features. Once drawn to a particular feature learners will have had more experience of responding to that feature than to any other and are more likely to continue to be drawn to it. Furthermore if no other feature presents itself as obvious, then responding to the initially noticed feature is a better option (highest level program law) than not responding at all, thus strengthening the attraction of this feature. This suggests that the way a learner responds to a task in their early experience of it, may have a significant effect on future activity. However, as the earlier discussion of metonymy revealed, learners might have no real choice about which feature to respond to hence it is likely that in a particular situation many learners may be drawn to a particularly dominant feature which they then persist with. This might account-for Davis' observation of different learners 'doing similar things'.

5.2.4 Implications for teaching and learning.

Acceptance of the conjecture that initial responses to a task may dominate future responses has implications for teaching and learning. If an initial response to a task, such as solving word problems, is inappropriate then by this argument a learner's future activity with problems may be dominated by inappropriate behaviour. However it does seem that learners do move on in their learning suggesting that either this conjecture is inaccurate or that learners encounter situations that cause them to develop. It would be helpful to teachers to be aware of the type of experiences that children need to have in order to counter the dominance of their early experiences. In the case of word problems, how can we help children to look beyond numbers and key words in problem statements and develop the type of desirable activity that Sowder (1988) suggests? This study suggests no 'magic wands' or 'quick fixes' but observations of the children over several months does throw light on approaches that may be fruitful in shaping children's attention and developing the nature of activity that they may engage in. In general it seems that tasks which have the effect of provoking instability (Siegler) or folding back (Kieren and Pirie) may be useful in bringing about change.

In general, arising from this study is the idea that long term immersion in appropriate tasks may be a productive way forward. If initial experiences are so dominating then it seems unlikely that some short term remedial approach will be able to break this grip on a learner's attention. As this study has shown it was only in the final session, six months after beginning the study, that Karen used subtraction rather than addition in response to any problem. Even if a learner does appear to have shifted their focus of attention or changed their behaviour as a result of a single experience, there is a strong possibility that there will be shifts back and forth between earlier and later tendencies before a particular area is mastered.

But what sort of tasks would it be fruitful for learners to become involved in? This study suggests that learners should be exposed to a range of associated tasks. So, if we are thinking of developing problem solving skills, then having word problems to solve may be an important task for children to work on. However, as discussed earlier, learners often see the goal of such a task as obtaining an answer rather than coming to understand the situation described by a problem, or understanding the mathematics related to it. Therefore, constant presentation of solving tasks may not provide sufficient stimulus to cause a shift in focus or a change in behaviour. The SMC tasks used in the study all involve word problems but, because of their specific nature each type of task appears to have a tendency to provoke a different type of activity. To some extent they tend to initiate asserting behaviour on behalf of the learners and provide situations where learners may be more likely to shift their noticing beyond the more superficial features of problem statements.

Hence, as an approach to developing general problem solving skills and to address issues of unstable behaviour, persistent behaviour and fluctuating attention, it is recommended that long term immersion in sorting, matching and constructing tasks may be a useful addition to conventional problem solving tasks and is worthy of further study.

6.0 Summary, Conclusions and Recommendations

6.0 Summary, Conclusions and Recommendations

The purpose of this chapter is to present a review of the conduct and findings of the study and discuss implications arising from it. The initial research questions will be discussed in the light of the data analysis, limitations of the approach to the study will be highlighted and recommendations for future areas of study will be identified.

My initial motivation had the goal of investigating 'using and applying' in the primary school with the intention of discovering something which might enhance children's experience and improve their performance. Opportunities to explore wider areas of application were difficult to secure due to the nature of the curriculum in schools that I had access to so the study came to focus on word problems. In particular I became interested in children's awareness of structural relationships embedded in word problems and how their thinking about these relationships might change over time and with exposure to sorting, matching and constructing tasks. To reveal useful information about this, the main focus of my study became addressing the following questions:

> What is the nature of the activity that learners engage in when responding to tasks that involve:
>
> - solving word problems,
> - sorting word problems,
> - matching word problems;
> - constructing word problems
>
> How can phenomena identified when analysing children's activity on these tasks be accounted-for in terms of relevant literature?
>
> What are the implications for teaching and learning arising from this analysis?

6.1 Review and critique of the methodology

The study involved working with a small group of children using tasks that focused on solving, sorting, matching and constructing word problems. Data relating to the children's behaviour and performance was collected in the form of their recordings, field notes, transcripts of discussion, and my post-session mental reconstruction of sessions. Accounts-of relevant incidents from my experiences outside of the study were also drawn on as data. Accounts-of whole sessions and incidents in sessions were constructed. The design of the study was in the spirit of the *Discipline of Noticing* and *grounded theory.*

There were several stages of data analysis. Initially incidents were noted, recorded and sometimes acted upon during sessions with the children. This information was also drawn upon, along with children's recording, transcripts and field notes, during the analysis of data that took place between sessions. During these stages of analysis common themes were noted and explored as were exceptions to themes which were noticed. Decisions about the conduct of subsequent sessions were influenced by the analysis that took place between sessions. The final stage of analysis took place after the final session. At this point the data was reviewed in a series of 'passes' in which particular themes were identified. The data was reorganised in various ways as a means of affording the possibility of attention being drawn to themes which were initially less obvious. Attempts were then made to account-for themes that had been identified. Where drawing on material in the initial literature review was inadequate in accounting-for phenomena other more relevant material was sought out.

There are several issues relating to the conduct of the study which need to be borne in mind when drawing on the conclusions and recommendations of the study. These are discussed below:

- The small sample size may affect the validity of generalising the findings of the study. However in drawing on my wider experiences as a learner, teacher and teacher-educator I have encountered similar phenomena to that evident in the study. Hence it seems likely that what I have noticed about the four children in the study is not limited to them.

- The sample is biased in the sense that the children come from the same year group of the same school and that three of the four children have English as an additional language. The gender breakdown of the group may also present a bias, as the two 'more able' children were boys and the two 'less-able' were girls.

- Some themes in the data were not identified until the final stage of analysis after the last session with the learners. To some extent it was possible to explore these themes further in the data that had already been collected. However due to the timing it was obviously not possible to design specific sessions in such a way as to facilitate the collection of data that may have better informed the theme that had been noticed.

- My aim in the study was to explore the potential of SMC tasks rather than exploring my teaching. Hence a decision was made to minimise my role in interactions with the children. In the analysis I did take account of interventions I made in the sense of decisions about which problem statements to use but I did not explore any potential effects of other aspects of my interactions with the children. These decisions were made with a view to making informed comment about the use of SMC tasks rather than making comment about my 'teaching'. However it is acknowledged that potential influences on the children have been ignored and these might have had some effect on learners' responses to the tasks.

- I am not an impartial observer although I may try to be. I bring to the study certain attunements. These attunements are likely to be different to the attunements of a different researcher. So I may see features that another observer 'misses' but also might 'miss' aspects that another is drawn to. I am also not the 'same' observer on different occasions. By this I mean that the sensitivities and attunements that I bring to a later session in the study have been shaped, sometimes significantly, by my experience of the earlier sessions. Hence the attunements that I bring to a later session are different to those I bring to an earlier one. Thus I am looking at the behaviour of the children in each of the sessions from different positions.

It may be possible to devise studies which minimise the effects of sample size and sample bias but the partiality of the researcher is always likely to be a significant aspect of research to consider, particularly in a qualitative study like this one. Once recognising this, the variation possible amongst observers could be considered as a useful tool in enhancing our understanding of research data. If I am open to considering different accounts-for my data that might be put forward by others then I am likely to develop a better understanding of the incidents that I am considering. Such a dialogue did take place in the course of the study. The data and my analysis of it was subjected to regular critique from both of my supervisors and in addition I have presented three seminars based on the study. Thus 'observers' with different attunements have commented on and hence influenced my analysis.

6.2 Findings of the study

In this section I will discuss how analysis of data collected in the study informs the original questions that I set out to answer. The first of my three questions was aimed at exploring the nature of activity that learners engage in when responding to tasks that involve:

- Solving word problems,
- Sorting word problems,
- Matching word problems,
- Constructing word problems,

Arising from analysis of the data collected across all of these tasks were two main emerging commonalties. These commonalities can be categorised to construct two contrasting phenomena, unstable behaviour and persistent behaviour.

Unstable behaviour.

This is the tendency for learners to display variation in the behaviour they engage in when responding to tasks that might suggest similar responses. This manifested itself in a number of ways in the study including:

- Different answers being given to the same task presented on different occasions;
- Different strategies being used with similar tasks such as related number and problem statements, presented on the same occasion;
- Learners attending to different foci in similar tasks on different occasions;

The striking nature of this behaviour was such that it frequently appeared on the surface that learners were regressing. For example:

- incorrect answers to questions were given despite the same question being answered correctly on a prior occasion;
- learners who had attended to detailed features of a situation on one occasion subsequently appeared unable to see beyond surface details on another;

The phenomenon of unstable behaviour was observed in all learners and appeared to be a significant feature of the activity that they engaged in during the sessions.

Persistent behaviour

This is the tendency for learners to display similar behaviour and attend to similar foci across tasks and over time despite this behaviour often being inappropriate to the situation. Particularly striking examples of persistent behaviour can be found in the observations of Karen and Barry. Karen consistently used addition when answering word problems and Barry appeared to 'invent' relationships between numbers rather than shift his attention to other features. Although this phenomenon contrasts with unstable behaviour, it too was evident in all of the learners. It was clear by the end of the study that shifts in attention and changes in behaviour had occurred during in the period though the striking feature was how slowly this process had taken place.

My second question focused on accounting–for the phenomena of unstable and persistent behaviour. A range of ideas evident in the literature have been explored in an attempt to address this question. A central idea is that of attention and awareness. What is important in any learner's activity on any task is what they are attending to and how (Mason, 2004). Mason suggests that a role of education is to develop an individual's "inner monitor" in order to influence attention when required. However the "inner monitor" is subject to interaction with "outer" influences. Some of these have a bearing on the focus of a learner's attention in the moment whilst others affect the development of the 'inner monitor'.

One way of examining these influences is to consider the features of the situation that a learner is working in and also the natural tendencies of the learner. It has already been explained that objects and the systems within which the objects are presented offer a range of affordances to a learner; and that these affordances are not equally weighted. Some will not be perceived by a learner while others will exert a strong influence. Affordances that are perceived, consciously or unconsciously, by a learner act as signifiers for action (Norman, 2007). Although they give a learner clues about how to proceed the role of signifiers is affected by factors that a learner brings to a situation.

A learner comes to situations with attunements. These will have developed from the natural powers, tendencies and ways of working of learners and from the effects of prior experience. Learners appear to have a tendency to initially be drawn to surface details of a situation and to use fast, automatic, effortless, unconscious and inflexible processes, such as metonymic triggers, typical of Leron and Hassan's System 1 (Leron and Hazzan, 2009) in moving forward. They also seem to have a disposition to attend to different features of a situation and respond in different ways (cognitive variability). It is perhaps this latter tendency that is useful in allowing a learner to learn and develop more appropriate strategies. In addition to this Davis (1984) has outlined some common principles of information processing that describe the ways that humans tend to 'work'. Of particular relevance are:

- Discriminations are only as fine as necessary;
- The permanence of productions;
- The highest-level programs must run;
- The undifferentiated binary-operation frame;

These mechanisms have presumably developed because they are useful in helping an individual 'make sense of' and respond appropriately to the world around them. However the nature of the 'world' presented in mathematics lessons has some features that interact with these natural mechanisms to produce inappropriate reactions to some situations. One such feature is the high frequency of tasks in some classrooms that present little difference between successive stimuli and require little variation of strategy in response. If there is little opportunity to discriminate then powers of discrimination are unlikely to develop. If variation of strategy is not required then the tendency for variation of strategy will be suppressed. Thus over time a learner may become very limited in the powers and strategies that they can bring to bear. Experiences such as these are unlikely to educate awareness and develop a learner's 'internal monitor' thus leaving the direction of the learner attention more dependent on the more prominent of the external influences.

However some classroom environments provide more open tasks requiring greater powers of discrimination and variation of thought and action. Experience of these may interact with a learner's natural powers to educate awareness and further enhance the natural powers. Environments like this may be successful in developing awareness to the level where a learner is able to suppress the tendency to draw on System 1 processes and begin to employ the slow, conscious, effortful and flexible processes of System 2 (Leron and Hazzan, 2009). For example a learner may come to ignore the pressures implied by a metonymic trigger and begin to use some form of metaphoric mapping strategy more typical of System 2.

These ideas will have a role in informing our understanding of the ways in which learners respond to all types of task but it is useful to consider explicitly how they might account-for the behaviour of the children in the study, specifically how they might account-for unstable behaviour and persistent behaviour.

Unstable behaviour was evident at different levels. For example it was clear that some children produced incorrect answers to problems that they had previously answered correctly. Less obvious was their tendency to shift the focus of their attention to different features on different occasions. Behaviour such as that in the former example I perhaps used to see as undesirable. However after considering this phenomena in more detail and reconciling it with the research and ideas of Seigler (2007) in his work on *cognitive variation*; and Keirnan and Pirie's (1994) concept of *folding back*, it seems evident to me that unstable behaviour is a necessary feature of the way that human beings are designed to work. Thus the unstable responses of children to tasks in the study could be a result of this natural approach, involving variation of thought and action, which encourages shifts of attention in order to more fully explore the tasks.

In accounting-for persistent behaviour it might be more fruitful to explore the attunements of learners that have developed as a result of their experiences of education. Seigler (2007) suggests that there is strong evidence that initial cognitive variability predicts subsequent learning. Yet as has been earlier highlighted many approaches to teaching mathematics tend to depress rather than encourage cognitive variation. For example children are often given exercises where they practice only a single operation; and in approaches to problem solving children are often presented with step-by-step methods rather than being encouraged to think for themselves. Such approaches tend not to provide the sort of experience necessary for the development of Mason's (2004) 'internal monitor'. Thus a learner's resources for resisting the strong draw on their attention made by 'prototypical' features such as numbers and keywords become diminished. Hence learners are likely to continually be drawn to these features. The longer that learners are presented with the type of task that limits cognitive variation and development of their 'internal monitor' the harder it becomes to initiate a shift from this persistent focus of attention and its related behaviour.

This analysis suggests that unstable behaviour is a natural and desirable consequence of the way that human beings are designed while persistent behaviour seems the product of some of the approaches to teaching that have perhaps dominated mathematics classrooms. Thus the account-for persistent behaviour given above seems to imply that some common approaches to 'school mathematics' might be disabling rather than enabling some pupils.

The final question considers the implications for teaching and learning arising from this study. This will involve the presentation of a number of recommendations including several designed to address the possible causes of inappropriate persistent behaviour. These are contained in the next section.

6.3 Recommendations

The focus of the study has been on working with word problems so several recommendations relate directly to this aspect of the mathematics curriculum. However more general recommendations about teaching and learning mathematics are made. It is unlikely that the adoption of any single recommendation alone will have significant effect on learning; the recommendations are inter-related and dependent on each other for their effects.

The nature of school mathematics needs to develop to involve a greater proportion of tasks that expose learners to opportunities for **long-term immersion in tasks**. This is required in order to give learners the opportunity to gain the potential benefits of cognitive variability and to encourage them to shift the focus of their attention beyond surface features of the situations they encounter. This might also minimise the effects that short-term strategies, such as the keyword approach to problem solving, seem to have on children's learning.

This is not meant to imply that learners should spend more time working on the type of tasks that they have always worked on. During this long-term immersion **learners should be exposed to a variety of related tasks**. They should work on the same or similar ideas but do so in different ways. In particular learners would benefit from working on a greater proportion of open tasks than is usually the case in mathematics lessons. On the theme of word problems learners should be exposed to sorting, matching and construction tasks as well ones that involve solving. The use of a variety of tasks is likely to promote cognitive variability and have an affect on the particular affordances to which a learner is drawn.

Within the range of tasks that a learner encounters there should be **opportunity to notice similarity, to discern difference and make decisions** on the basis of this. Asserting behaviour on the part of learners needs to be encouraged. Open-ended approaches to sorting and matching of appropriately selected sets of word problems may afford possibilities of noticing and discernment that are not evident in solving tasks.

There should be **an emphasis on learners working on tasks rather than through them** in order to engage in the type of behaviour that is likely to promote shifts of attention and development of awareness. In particular opportunities for appropriate variation of strategy and 'folding back' should be encouraged.

As the tasks that a learner works on are either chosen or designed by teachers and the ways that learners work on them are strongly influenced by teachers these recommendations have important **implications for the development of teacher subject and pedagogical knowledge.**

In order to appropriately guide learners in working with (addition and subtraction) word problems teachers need to develop an awareness-in-discipline that reflects the different models of addition and subtraction and the mathematical structures on which the problems may be based. In particular they need to be aware of Haylock's (2006) structures and Riley et al's. (1983) word problem categories. Only with this knowledge are they likely to design effective tasks with word problems and to subsequently make appropriate intervention in response to learner activity.

In addition a teacher needs to be aware of how a learner behaves, what they attend to, how they attend and how particular experiences affect the development of a learner's natural powers. In short they need to develop appropriate awareness-in-counsel. Here it may be useful to discuss the models of learning that teachers may possess. Many teachers act in ways that suggest that they see learning as a one-way process, a ladder metaphor where learning is reflected as steps up the ladder. Steps down the ladder are considered as regression and many attempts to account-for this negative progress may result in blame being attributed to learner or teacher. This metaphor is strongly suggested by current approaches to teaching and learning evident in many English schools. This model is in contrast to that implied by this study where learning is seen as a non-unidirectional process where the manifestations of cognitive variability and 'folding back' are natural and necessary features. Thus it is important that a teacher views learning in such a way that they see 'steps back' as opportunities rather than as a lack of progression. To build on these opportunities teachers need to be aware of where a learner's attention may be focused and what they might do to encourage some shift in this focus. To take a relevant example, it would be useful for a teacher to be aware that Karen is attending to the numbers when solving problems and that use of a sorting or matching task may have some benefits in altering this focus. Awareness that this process may be slow is also necessary.

6.5 Future research

Below I offer some suggestions for future research based on my review of this particular study:

The tasks utilized in this study drew on only a small range of the possible variations in structure and feature of addition and subtraction word problems. Hence an obvious recommendation would be to develop studies that used SMC tasks focusing on a wider range of structure and feature of addition and subtraction word problem. Use of multiplication and division word problems may also be appropriate.

Sorting and matching tasks each tended to draw a learner's attention to different features of a situation. They seemed to offer slightly different affordances to each other. Exploration of the affordances offered by sorting and matching tasks in other areas of the mathematics curriculum would thus seem a rich vein to explore.

A key aspect of this study has been the attempt to identify features of a situation that learners are attending to and how they are attending. Attention does not only relate to working with word problems. Thus it may be fruitful to devise studies focused on other areas of mathematics that are designed to attempt to identify the focus of attention and the nature of a learner's subsequent activity. This could be carried out in a study that involves withdrawal from normal teaching arrangements rather like this study. However the identification of focus and nature of attention in a *normal* teaching situation might reveal particularly interesting and useful information.

6.6 Personal development

At various points in this thesis I have indicated that this study has been a vehicle for stimulating and identifying my development in addition to learning about that of the children. I do feel that I have learnt much about how the children respond to word problems and how their attention and awareness fluctuates and develops over time. In considering this I have also explored areas that I previously knew little or nothing about. In particular I am thinking about metonymy, prototypes, affordances and so on. If I had encountered these ideas prior to the study I doubt that I would have seen their relevance to mathematics education. However just as SMC tasks afforded possibilities to learners that seemed absent from solving tasks, this study has afforded me the opportunity to learn about these ideas and to see them in ways that I doubt I would have achieved in any other way.

Working with the study over the last six years has gradually developed the way I see things. I now seem sensitized to features that I previously ignored, I seem to be able to draw distinctions where previously I didn't but perhaps most significant of all I approach any situation with the view that there is much to be noticed and that however much I notice in a situation I realise there is always yet more to see. Hence, though this is the end of this thesis, it is just one step in my development... perhaps the first of many in a new direction.

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Appendices

Appendix 1

<u>An example of an 'Answer Frame'</u>.

Solving Word Problems

THE PROBLEM

USEFUL INFO.

USELESS INFO.

CALCULATION NEEDED

ANSWER TO CALCULATION

ANSWER TO THE PROBLEM

Appendix 2

Change Problems

Type	Example	Schema	Direction	Unknown
Change 1	Joe had 3 marbles; then Tom gave him 5 more marbles; how many marbles does Joe have now?	change	increase	result set
Change 2	Joe had 8 marbles; then he gave 5 marbles to Tom; how many marbles does Joe have now?	change	decrease	result set
Change 3	Joe had 3 marbles; then Tom gave him some more marbles; now Joe has 8 marbles; how many marbles did Tom give him?	change	increase	change set
Change 4	Joe had 8 marbles; then he gave some marbles to Tom; now Joe has 3 marbles; how many marbles did he give to Tom?	change	decrease	change set
Change 5	Joe had some marbles; then Tom gave him 5 more marbles; now Joe has 8 marbles; how many marbles did Joe have in the beginning?	change	increase	start set
Change 6	Joe had some marbles; then he gave 5 marbles to Tom; how many marbles did Joe have in the beginning?	change	decrease	start set

Appendix 3

Combine Problems

Type	Example	Schema	Direction	unknown
Combine 1	Joe has 3 marbles; Tom has 5 marbles; how many marbles do they have altogether?	combine	-	superset
Combine 2	Joe and Tom have 8 marbles altogether; Joe has 3 marbles; how many marbles does Tom have?	combine		subset

Appendix 4

Compare Problems

Type	Example	Schema	direction	unknown
Compare 1	Joe has 8 marbles; Tom has 5 marbles; how many marbles does Joe have more than Tom?	compare	more	Difference set
Compare 2	Joe has 8 marbles; Tom has 5 marbles; how many marbles does Tom have less than Joe?	compare	less	Difference set
Compare 3	Joe has 3 marbles; Tom has 5 more marbles than Joe; how many marbles does Tom have?	compare	more	Compared set
Compare 4	Joe has 8 marbles; Tom has 5 marbles less than Joe; how many marbles does Tom have?	compare	less	Compared set
Compare 5	Joe has 8 marbles; he has 5 more marbles than Tom; how many marbles does Tom have?	compare	more	Reference set
Compare 6	Joes has 3 marbles; he has 5 marbles less than Tom; how many marbles does Tom have?	compare	less	Reference set

Appendix 5

Initial Probe Problem Statements

Probe problem A: Change 1 Time ordered sequence Shahina has £16. She is given £25 more by her parents. How much does she have now?
Probe problem B: Change 1 Altered sequence Before he was given £23 by his parents Leroy had £19. How much does he have now?
Probe problem C: Change 6 Temporal sequence On Monday morning all the children in Class 2B are present. Six of the children go home ill. There are 28 children left in the class. How many children were present in the morning?
Probe problem D: Change 6 Altered sequence There are 27 children left in the class at home time. During the day 5 children had gone home because they were ill. How many children were present at the beginning of the day?
Probe problem E: Combine 2 Explicit There are 32 children in Class 3C. There are 15 boys in the class. How many girls are there?
Probe problem F: Combine 2 Reordered There are 16 boys in a class of 31 children. How many girls are there?
Probe problem G: Nonsense Class 4D has 16 boys and 17 girls. How old is the teacher?

Initial Probe Number Statements

A	$16 + 25 = \square$
B	$23 + 19 = \square$
C	$\square - 6 = 28$
D	$\square - 5 = 27$
E	$15 + \square = 32$
F	$16 + \square = 31$
G	$16 + 17 = \square$

Appendix 6

Problem Set

BusInc1
A
There were 27 passengers on a bus. At the bus stop another 14 passengers got on. How many passengers were on the bus now?

BusDec1
B
On a bus there were 32 passengers. At the bus stop 13 passengers got off. How many passengers were on the bus now?

BusInc2
C
On a bus there were 35 people. At the bus stop some passengers got on. There were now 49 passengers on the bus. How many people got on the bus at the bus stop?

BusDec2
D
There were 34 people on a bus. At the bus stop some people got off. There were 19 passengers left on the bus. How many passengers got off the bus at the bus stop?

BusInc3
E
There were some passengers on a bus. At the bus stop another 15 passengers got on the bus. Now there are 33 people on the bus. How many passengers were on the bus before it stopped?

BusDec3
F
On a bus there were some people. At the bus stop 16 passengers got off. Now there were 22 people on the bus. How many passengers were on the bus before it stopped?

MonInc1
G
Paul had £27. His parents gave him £16 for his birthday. How much money does he have now?

MonDec1
H
Alan had £21 for his birthday. He spent £6 on a model. How much money does he have now?

MonInc2
I

Ann had £28. Her parents gave her some money for her birthday. Now she has £41. How much money was she given for her birthday?

MonDec2
J
Asif had £41. He gave some of the money to his sister. He had £28 left. How much money did he give to his sister?

MonInc3
K
Sue had some money. On her birthday she was given £16. Now she has £31. How much did she have before her birthday?

MonDec3
L
Harry had some money. He spent £12 on a birthday present for his sister and has £16 left. How much did he have before buying the present?

BusInc1a
M
At a bus stop 17 passengers got on a bus that was already carrying 26 passengers. How many people were on the bus now?

BusInc2a
O
At a bus stop some people got on a bus that was already carrying 28 passengers. There are now 41 people on the bus. How many people got on the bus at the bus stop?

MonInc1a
S
Asif's parents gave him £12 for his birthday. He already had £39. How much money does he have now?

MonInc2a
U
Ann's parents gave her some money for her birthday. She already had £25. Now she has £41. How much money was she given for her birthday?

MonInc3a
W
Sue had £27 after she had been given £12 by mum. How much money did she have before mum gave her the money?

Appendix 7

Initial / Final Probe Comparisons

Karen		February		July	
Number statement	**Problem statement**	**Number statements**	**Problem statements**	**Number statements**	**Problem statements**
16 + 25= □	**A Change 1** Shahina has £16. She is given £25 more by her parents. How much does she have now?	41	£31 20+10=30 6+5=11	41	36 She has £36.00
23 + 19 = □	**B Change 1** Before he was given £23 by his parents Leroy had £19. How much does he have now?	32	32 20+10=30 9+3=12	42	42 Leroy has £42
□- 6 = 28	**C Change 6** On Monday morning all the children in Class 2B are present. Six of the children go home ill. There are 28 children left in the class. How many children were present in the morning?	47	Can't do it	34	Don't know
□ – 5 = 27	**D Change 6** There are 27 children left in the class at home time. During the day 5 children had gone home because they were ill. How many children were present at the beginning of the day?	31	32 27 + 5 =32	32	22 There are 22 people where present at beginning
15 + □ =32	**E Combine 2** There are 32 children in Class 3C. There are 15 boys in the class. How many girls are there?	22	47 30 + 10=40 5 + 2 = 7	17	17 There 17 girls in the class.
16 + □ = 31	**F Combine 2** There are 16 boys in a class of 31 children. How many girls are there?	47	Can't not do it	15	15 There are 15 girls in the class
16 + 17 =	**G Nonsense** Class 4D has 16 boys and 17 girls. How old is the teacher?	Can't do it	This is a silly question	32	Don't know

Haleema		February		July	
		Number statements	Problem statements	Number statements	Problem statements
16 + 25= □	**A Change 1** Shahina has £16. She is given £25 more by her parents. How much does she have now?	41	41 £16+£25 T 10+20=30 U 6+5=11	41	40 Shahina has £40 mor altogether.
23 + 19 = □	**B Change 1** Before he was given £23 by his parents Leroy had £19. How much does he have now?	42	42	42	55 He has £55 now.
□- 6 = 28	**C Change 6** On Monday morning all the children in Class 2B are present. Six of the children go home ill. There are 28 children left in the class. How many children were present in the morning?	22	*No response*	34	34 In the present there where 34 children
□ – 5 = 27	**D Change 6** There are 27 children left in the class at home time. During the day 5 children had gone home because they were ill. How many children were present at the beginning of the day?	12	I can't do it	33	In the beginning there where 32 children in the class
15 + □ =32	**E Combine 2** There are 32 children in Class 3C. There are 15 boys in the class. How many girls are there?	22	17	18	Hard
16 + □ = 31	**F Combine 2** There are 16 boys in a class of 31 children. How many girls are there?	15	There are 15 girls.	15	There are 15 girls in the classroom.
16 + 17 =	**G Nonsense** Class 4D has 16 boys and 17 girls. How old is the teacher?	33	I can't tell and it is a silly question.	32	I thin the teacher is 33 years old

Alan		February		July	
		Calculation	Problem	Calculation	Problem
16 + 25= □	**A Change 1** Shahina has £16. She is given £25 more by her parents. How much does she have now?	41	41 £16 + 25 =41	41	41 £16+£25=£41 Shahina has £41
23 + 19 = □	**B Change 1** Before he was given £23 by his parents Leroy had £19. How much does he have now?	42	44 £19 + £23 =44	42	42 23+19=41
□- 6 = 28	**C Change 6** On Monday morning all the children in Class 2B are present. Six of the children go home ill. There are 28 children left in the class. How many children were present in the morning?	34	34	34	?
□ – 5 = 27	**D Change 6** There are 27 children left in the class at home time. During the day 5 children had gone home because they were ill. How many children were present at the beginning of the day?	32	22 27-5=22	32	32 27+5=32
15 + □ =32	**E Combine 2** There are 32 children in Class 3C. There are 15 boys in the class. How many girls are there?	17	17	26	17 32-15=17 There are 17 girls in 3C
16 + □ = 31	**F Combine 2** There are 16 boys in a class of 31 children. How many girls are there?	15	15	15	15 16-31=15
16 + 17 =	**G Nonsense** Class 4D has 16 boys and 17 girls. How old is the teacher?	33	I don't know how old the teacher	33	I can't tell.

Barry		February		July	
Number statement	Problem Statement	Number statement	Problem Statement	Number statement	Problem Statement
16 + 25= □	**A Change 1** Shahina has £16. She is given £25 more by her parents. How much does she have now?	41 *10+20=30* *5+6=11*	41 *10 + 20=30* *6+5=11* *She has got £41*	31	41 *16+25=41*
23 + 19 = □	**B Change 1** Before he was given £23 by his parents Leroy had £19. How much does he have now?	42	42 *10+20=30* *3+9=12* *£42.00*	42	32 *19+23=32*
□- 6 = 28	**C Change 6** On Monday morning all the children in Class 2B are present. Six of the children go home ill. There are 28 children left in the class. How many children were present in the morning?	34	34 *28+6=34*	32	34 *28+6=34*
□ – 5 = 27	**D Change 6** There are 27 children left in the class at home time. During the day 5 children had gone home because they were ill. How many children were present at the beginning of the day?	32	39 *27+5=12* *27+12=39*	32	32 *27+5=32*
15 + □ =32	**E Combine 2** There are 32 children in Class 3C. There are 15 boys in the class. How many girls are there?	17	17 *32-15=17*	26	17 *15+17=32*
16 + □ = 31	**F Combine 2** There are 16 boys in a class of 31 children. How many girls are there?	15	25 *30-10=20* *6-1=5* *20+5=25 girls*	25	15 *16+15=31*
16 + 17 =	**I Nonsense** Class 4D has 16 boys and 17 girls. How old is the teacher?	33	33 *10+10=20* *6+7=13* *33 years old*	33	*"out of order"*

Appendix 8

Example of Data.

Transcript from March Session

Barry 24th March

BusDec3 F
On a bus there were some people. At the bus stop 16 passengers got off. Now there were 22 people on the bus. How many passengers were on the bus before it stopped?

B I've finished mine.

JB ... OK. You had F, and what's your answer to F?

B 22 add 16 = and then we put the answer, I put 22 and 8 = 30, 30 + xxxxx, then I put 30 + 8 = 38 xxxx

JB So the answer's 38.

B Mm.

JB So 38 what?

B 38 passengers.

JB 38 passengers. Where were the passengers?

B On the bus.

JB OK, 38 passengers on the bus. Is this before... it got to the bus stop or...

B Before.

JB Before, OK. So you've added. How did you know to add?

B Its easy, you just do counting up xxxx

JB But how did you know that you had to do adding, to answer the problem? How did you know it wasn't take away or multiplying.

B Because these passengers got off and it said how many passengers were on the bus before it, from the past, present, I had to add it on.

JB	OK. Tell me that again.
B	There was ... some people on the bus, the bus stops, 16 passengers got up and there was 22 people left on the bus, and then it told me how many people were there before, I had to add 22 and 16 to get the right answer.
JB	OK. Right, that's good.
B	It's like the 16 other passengers got back on the bus.
JB	Right, so to find out how many were on, you sort of in your mind had to, in the first place you had to put the passengers back on the bus, in your mind.
B	Yes.

Sorting

Barry was presented with six change problems to sort into categories of his choice. Three of the problems were 'increasing' with the unknown in each in a different position. Three were 'decreasing' again with an unknown in each of the three possible positions. (? +/- ? = ?).

BusInc1 **A** There were 27 passengers on a bus. At the bus stop another 14 passengers got on. How many passengers were on the bus now?
BusDec1 **B** On a bus there were 32 passengers. At the bus stop 13 passengers got off. How many passengers were on the bus now?
BusInc2 **C** On a bus there were 35 people. At the bus stop some passengers got on. There were now 49 passengers on the bus. How many people got on the bus at the bus stop?
BusDec2 **D** There were 34 people on a bus. At the bus stop some people got off. There were 19 passengers left on the bus. How many passengers got off the bus at the bus stop?
BusInc3 **E** There were some passengers on a bus. At the bus stop another 15 passengers got on the bus. Now there are 33 people on the bus. How many passengers were on the bus before it stopped?
BusDec3 **F** On a bus there were some people. At the bus stop 16 passengers got off. Now there were 22 people on the bus. How many passengers were on the bus before it stopped?

B Yes.

JB So, now lets have a look, what have you done? Are those three together?

B Yes, because A's got, those are xxxx F. So swap that around.

JB OK. So before you tell me about it, can you tell me how you've sorted it. So are you saying that F B and A are in one group.

B Yes.

JB OK. So we've got F, B and A.. And you're very particular about that order, why?

B Yes. Because 22 add a 10 equals 32, then take away another number equals that. They're the 20s.

JB So you're saying 22 add 10 makes 32 in B and 32 take away 5 makes 27. That's how you've linked them together. OK. Anything else you want to tell me about why you've put them together?

B Because these are all 10s. 16, 13, 14.

JB OK, because the numbers are all tens. OK. So have you put those together or haven't you looked at those yet?

B I put these together because this one

JB So that's E, D and F, sorry E, D and C you've put together.

B I put it all because there's 33, then it goes 34, then 35.

JB Right, so its 33, 34 and 35 are the three numbers. Brilliant. OK, that's good. So, and you've not only put them in groups but you've given me an order as well. That's very good.

...

Barry is then asked to sort the problems a different way.

> ... I'm going to ask you to, this time to do something that might be quite hard. When you're looking at them to sort them, try not to use the numbers. Try and look for something in them other than the numbers, OK? Does that make sense?

B Yes.

...

JB OK, have a few minutes to think about that. What have you done?

... Then I want you to have a look see if you can think of a different way or sorting them? OK? It might not be easy, but think of a different way. What have you done this time?

B All of them together.

JB You put them all together?

B Yes.

JB Why?

B Because they're all about a bus and passengers.

JB OK, that's a good idea.

B Its about, and its about adding passengers and taking away passengers.

JB Right. ....So are all of them about adding or taking away passengers, or are only some of them?

Some of them.

JB Right. Could I make a suggestion to you based on what you've just told me? Could you put them together into the ones that are adding passengers and the ones that are taking away passengers and make your two groups that way, and I'll come back to you in a moment.

One last thing. Could you write the letters out and could you not do the sums but could you write down the sum that you need to do to solve each one, so if I put G there, what would you have to do to find G. Not all the working out, but just the sum, or the calculation. Right, what have you done here? These are sorted according to

B B, D and F are people that are getting off.

JB So B, D and F are

B Passengers getting off because they're got off, got off, got off. Triple got offs.

JB Triple got offs.

B Look got off, got off, got off.

JB Got off, got off, got off, so its a triple got of, I like that. And so let me guess, these are triple

B These are only, there's one got on and double on off, on off. Passengers on off, passengers on off.

JB OK, OK. Good. So this is about a bus where people get off the bus and here is a bit about people getting on the bus. Good. Right, which lot would you like to look at, would you like to look at those or those, I don't mind. OK, we'll look at B, D and F. OK. You've told me that B, D and F are the same, or they go together because they're all about getting off a bus. I want you to have a look at those for a minute, what's different about them? Are they different to each other in some way? Just have a think for a couple of minutes.

B I've found out that them two are the same.

JB OK, why are they the same?

B They're both adding. Because you can add

JB OK, so its to do with changing the amount of money so it can add, it can take away. OK, I wonder if you could look at these three problem with me for Barry, we both look at the same ones. Barry has put these together, and I'll give you a chance to read them, he's put them together because they are "got off" problems, he's called them triple got-off problems, which is quite a nice description I like that. So, what I've asked him to do is to look at those problems and I want to know what's different about each one, is there something different.

B I've found it, I've found it already.

JB OK, explain to Alan.

B Look, there's a question, they're different endings, we're on the bus now, the bus stop, bus before it stopped.

JB OK, right.

B They're both the same.

A Yeah, but they're different sayings, like the bus stopped, the bus before it stopped...

B Its the same meanings.

JB So are you just saying that there are different words there? You're telling me something about the meanings Alan.

B Yes.

...Interruption 30secs...

JB OK, I'm interested that you said meanings.

B Yes, the meanings are the same but they're saying it different.

Writing a problem

Barry was asked to write a 'number sentence' to go with problem B.

BusDec1
B
On a bus there were 32 passengers. At the bus stop 13 passengers got off. How many passengers were on the bus now?

Barry wrote a 'copy' of problem B and then wrote a number sentence:

A bus had 32 passengers. The bus stopped at a bus stop. 13 passengers walked off. How many passengers now?

$$32 - 13 = 21$$

And the working out:

30-10=20
3-2= 1

JB	.... Barry, I only wanted you to do something like that, OK?
B	I don't want to do the sentence first....
JB	OK, but there's no need to copy it we can just write the letter
B	No, I'm not copying it, because look.
JB	Oh right, sorry.
B	I've said 32 passengers
A	Like notes to help him.
B	The bus stopped at a bus stop 13 passengers walked off, how many passengers now?
JB	So you've changed the order of things in the problem, which problem was it, B? To help you sort it out, I do apologise, I'm sorry, that's excellent.

Appendix 9

Vergnaud's Categories of relationship in addition and subtraction situations

Category 1: Composition of two measures

Example:
Peter has 6 marbles in his right-hand pocket and 8 marbles in his left-hand pocket. He has 14 marbles altogether.

Category 2: A transformation links two measures (state-transformation-state)

Example:
Peter had 17 marbles before playing. He lost 4 marbles. He now has 13 marbles.

Category 3: A static relationship links two measures (state-relationship-state)

Example:
Peter has 8 marbles. He has 5 more than John. John has 3 marbles.

Category 4: Composition of two transformations (transformation-transformation-transformation)

Example:
Peter won 6 marbles in the morning. He lost nine marbles in the afternoon. Altogether he lost 3 marbles.

Category 5: A transformation that links two static relationships (relationship – transformation- relationship)

Example:
Peter owed Henry 6 marbles. He gives him 4. He still owes Henry 2 marbles.

Category 6: Composition of two static relationships (relationship-relationship-relationship)

Examples:
Peter owes 8 marbles to Henry, but Henry owes 6 to Peter. So Peter owes 2 marbles to Henry.
Robert has 7 marbles more than Susan. Susan has 3 marbles less than Connie. Robert has 4 marbles more than Connie

(Vergnaud, 1982)